Review for Exam 3.

- Sections 15.1-15.4, 15.6.
- 50 minutes.
- 5 problems, similar to homework problems.
- No calculators, no notes, no books, no phones.
- No green book needed.

Triple integral in spherical coordinates (Sect. 15.6).

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

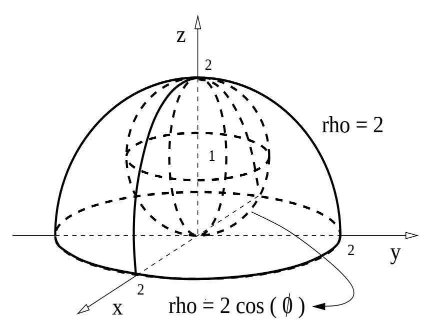
Solution: First sketch the integration region.

- $\rho = 2\cos(\phi)$ is a sphere, since

$$\rho^2 = 2\rho\cos(\phi) \Leftrightarrow x^2 + y^2 + z^2 = 2z$$

$$x^2 + y^2 + (z-1)^2 = 1.$$

- $\rho = 2$ is a sphere radius 2 and $\phi \in [0, \pi/2]$ says we only consider the upper half of the sphere.

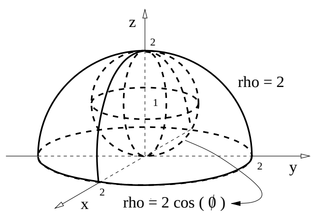


Triple integral in spherical coordinates (Sect. 15.6).

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution:



$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

$$V = 2\pi \int_0^{\pi/2} \Big(\frac{\rho^3}{3}\Big|_{2\cos(\phi)}^{2} \Big) \sin(\phi)\, d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/2} \Big[8\sin(\phi) - 8\cos^3(\phi)\,\sin(\phi) \Big]\, d\phi.$$

$$V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi)\Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$$

Triple integral in spherical coordinates (Sect. 15.6).

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: $V = \dfrac{16\pi}{3} \Big[\Big(-\cos(\phi)\Big|_0^{\pi/2} \Big) - \displaystyle\int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

$$V = \frac{16\pi}{3} \Big[1 + \int_1^0 u^3\, du \Big] = \frac{16\pi}{3} \Big[1 + \Big(\frac{u^4}{4}\Big|_1^0 \Big) \Big] = \frac{16\pi}{3} \Big(1 - \frac{1}{4} \Big).$$

$$V = \frac{16\pi}{3}\,\frac{3}{4} \quad \Rightarrow \quad V = 4\pi.$$

◁

Triple integral in cylindrical coordinates (Sect. 15.6).

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

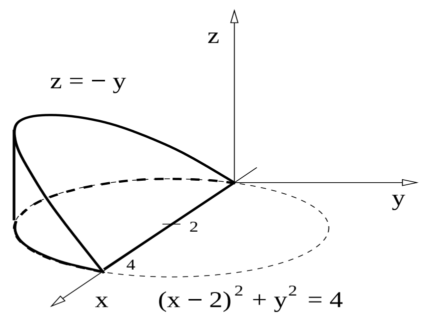
Solution: First sketch the integration region.

- $(x-2)^2 + y^2 = 4$ is a circle, since

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r\cos(\theta)$$

$$r = 4\cos(\theta).$$

- Since $0 \leqslant z \leqslant -y$, the integration region is on the $y \leqslant 0$ part of the $z = 0$ plane.



Triple integral in cylindrical coordinates (Sect. 15.6).

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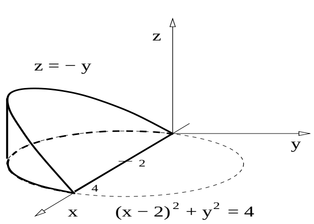
Solution:



$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \int_0^{-r\sin(\theta)} r\,dz\,dr\,d\theta.$$

$$V = \int_{3\pi/2}^{2\pi} \int_0^{4\cos(\theta)} \bigl[-r\sin(\theta) - 0\bigr]\,r\,dr\,d\theta$$

$$V = -\int_{3\pi/2}^{2\pi} \Bigl(\frac{r^3}{3}\Big|_0^{4\cos(\theta)}\Bigr)\sin(\theta)\,d\theta.$$

$$V = -\int_{3\pi/2}^{2\pi} \frac{4^3}{3}\cos^3(\theta)\sin(\theta)\,d\theta.$$

Triple integral in cylindrical coordinates (Sect. 15.6).

Example

Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution: $V = -\displaystyle\int_{3\pi/2}^{2\pi} \frac{4^3}{3} \cos^3(\theta)\sin(\theta)\, d\theta.$

Introduce the substitution: $u = \cos(\theta)$, $du = -\sin(\theta)\, d\theta$;

$$V = \frac{4^3}{3}\int_0^1 u^3\, du = \frac{4^3}{3}\Big(\frac{u^4}{4}\Big|_0^1\Big) = \frac{4^3}{3}\,\frac{1}{4}.$$

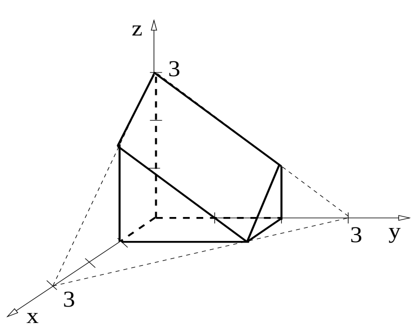
We conclude: $V = \dfrac{16}{3}$. $\triangleleft$

Triple integral in Cartesian coordinates (Sect. 15.4).

Example

Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 2$ and $0 \leqslant x \leqslant 1$, while the top side lies in the plane $x + y + z = 3$.

Solution:



$$V = \int_0^1\int_0^2\int_0^{3-x-y} dz\, dy\, dx,$$

$$V = \int_0^1\int_0^2 (3-x-y)\, dy\, dx,$$

$$= \int_0^1 \Big[(3-x)\Big(y\Big|_0^2\Big) - \frac{1}{2}\Big(y^2\Big|_0^2\Big)\Big]\, dx,$$

$$V = \int_0^1 \Big[2(3-x) - \frac{4}{2}\Big]\, dx.$$

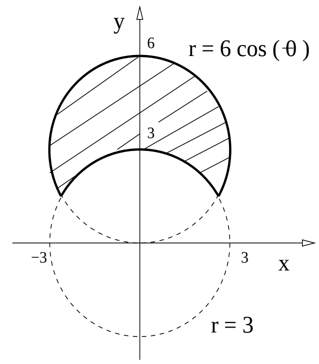
$$V = \int_0^1 (4-2x)\, dx = \Big[4\Big(x\Big|_0^1\Big) - \Big(x^2\Big|_0^1\Big)\Big] = 4 - 1 \quad\Rightarrow\quad V = 3.$$

Double integrals in polar coordinates. (Sect. 15.3)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Solution: First sketch the integration region.



- $r = 6\sin(\theta)$ is a circle, since

$$r^2 = 6r\sin(\theta) \Leftrightarrow x^2 + y^2 = 6y$$

$$x^2 + (y-3)^2 = 3^2.$$

- The other curve is a circle $r = 3$ centered at the origin.

The condition $3 = r = 6\sin(\theta)$ determines the range in θ. Since $\sin(\theta) = 1/2$, we get $\theta_1 = 5\pi/6$ and $\theta_0 = \pi/6$.

Double integrals in polar coordinates. (Sect. 15.3)

Example

Find the area of the region in the plane inside the curve $r = 6\sin(\theta)$ and outside the circle $r = 3$, where r, θ are polar coordinates in the plane.

Solution: Recall: $\theta \in [\pi/6, 5\pi/6]$.

$$A = \int_{\pi/6}^{5\pi/6} \int_3^{6\sin(\theta)} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \Big(\frac{r^2}{2}\Big|_3^{6\sin(\theta)} \Big)\,d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2}\sin^2(\theta) - \frac{3^2}{2}\Big]\,d\theta = \int_{\pi/6}^{5\pi/6} \Big[\frac{6^2}{2^2}\big(1 - \cos(2\theta)\big) - \frac{3^2}{2}\Big]\,d\theta$$

$$A = 3^2\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big) - \frac{3^2}{2}\Big(\sin(2\theta)\Big|_{\pi/6}^{5\pi/6}\Big) - \frac{3^2}{2}\Big(\frac{5\pi}{6} - \frac{\pi}{6}\Big).$$

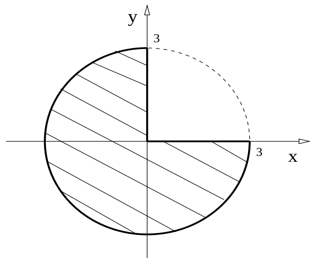
$$A = 6\pi - 3\pi - \frac{3^2}{2}\Big(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\Big), \text{ hence } A = 3\pi + 9\sqrt{3}/2. \quad \triangleleft$$

Double integrals in Cartesian coordinates. (Sect. 15.2)

Example

Find the y-component of the centroid vector in Cartesian coordinates in the plane of the region given by the disk $x^2 + y^2 \leqslant 9$ minus the first quadrant.

Solution: First sketch the integration region.



$\overline{y} = \dfrac{1}{A} \iint_R y\, dA$, where $A = \pi R^2 (3/4)$, with $R = 3$. That is, $A = 27\pi/4$. We use polar coordinates to compute $\overline{y}$.

$$\overline{y} = \frac{4}{27\pi} \int_{\pi/2}^{2\pi} \int_0^3 r \sin(\theta)\, r dr\, d\theta.$$

$$\overline{y} = \frac{4}{27\pi} \Big(-\cos(\theta) \Big|_{\pi/2}^{2\pi} \Big) \Big(\frac{r^3}{3} \Big|_0^3 \Big) = \frac{4}{27\pi} (-1)(9) \quad \Rightarrow \quad \overline{y} = -\frac{4}{3\pi}.$$

Double integrals in polar coordinates. (Sect. 15.2)

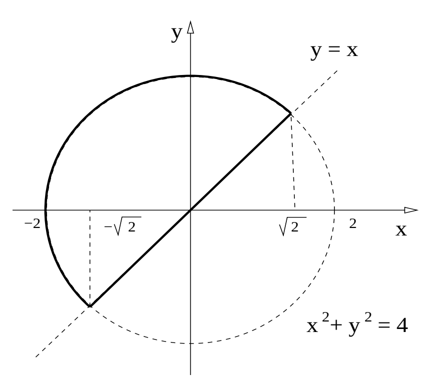
Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{-\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \big(x^2 + y^2\big)\, dy\, dx.$$

Solution: First sketch the integration region.

- $x \in [-2, \sqrt{2}]$.
- For $x \in [-2, -\sqrt{2}]$, we have $|y| \leqslant \sqrt{4 - x^2}$, so the curve is part of the circle $x^2 + y^2 = 4$.
- For $x \in [-\sqrt{2}, \sqrt{2}]$, we have that y is between the line $y = x$ and the upper side of the circle $x^2 + y^2 = 4$.



Double integrals in polar coordinates. (Sect. 15.2)

Example

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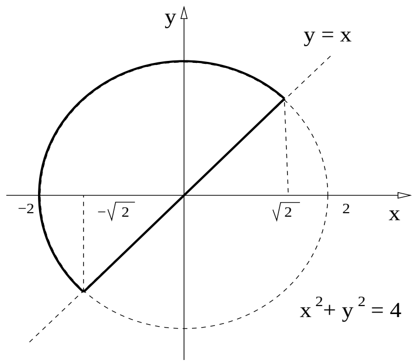
$$I = \int_{-2}^{-\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution:



$$I = \int_{\pi/4}^{5\pi/4} \int_0^2 r^2\, r dr\, d\theta$$

$$I = \Big(\frac{5\pi}{4} - \frac{\pi}{4}\Big) \int_0^2 r^3\, dr$$

$$I = \pi\Big(\frac{r^4}{4}\Big|_0^2\Big)$$

We conclude: $I = 4\pi$. $\triangleleft$

Double integrals in polar coordinates. (Sect. 15.2)

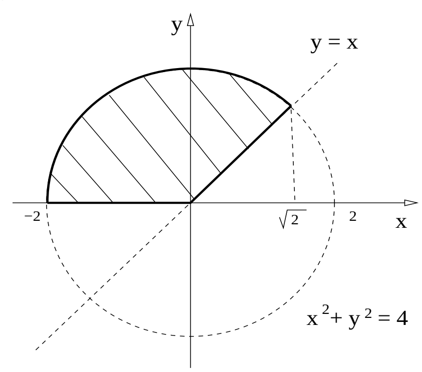
Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx + \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$$

Solution: First sketch the integration region.

- $x \in [-2, \sqrt{2}]$.
- For $x \in [-2, 0]$, we have $0 \leqslant y$ and $y \leqslant \sqrt{4 - x^2}$. The latter curve is part of the circle $x^2 + y^2 = 4$.
- For $x \in [0, \sqrt{2}]$, we have $x \leqslant y$ and $y \leqslant \sqrt{4 - x^2}$.



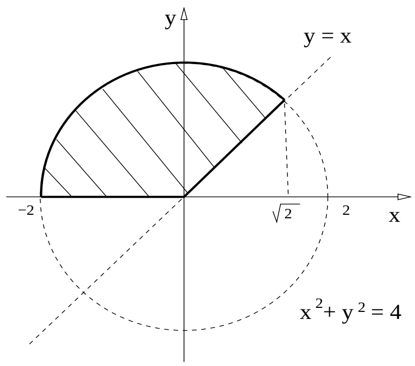
Double integrals in polar coordinates. (Sect. 15.2)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} \left(x^2 + y^2\right) dy\, dx + \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} \left(x^2 + y^2\right) dy\, dx$$

Solution:



$$I = \int_{\pi/4}^{\pi} \int_{0}^{2} r^2\, r dr\, d\theta$$

$$I = \frac{3\pi}{4} \left(\frac{r^4}{4}\Big|_0^2 \right)$$

We conclude: $I = 3\pi$. $\triangleleft$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- Mass and center of mass of wires.

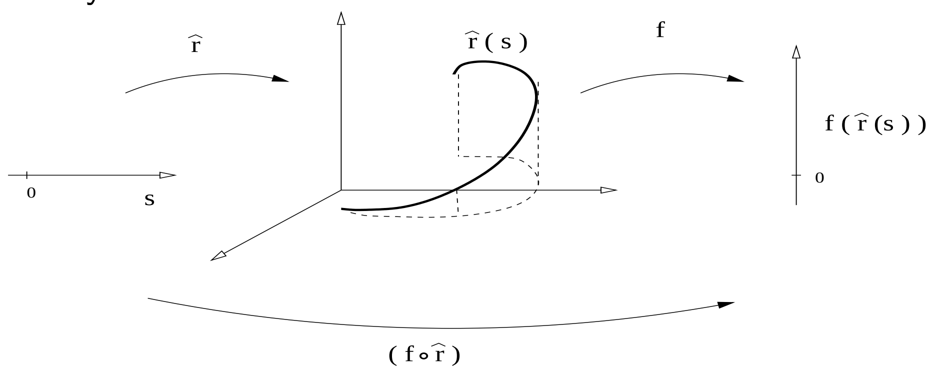
Line integrals in space.

Definition

The *line integral* of a function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along a curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C f\, ds = \int_{s_0}^{s_1} f\big(\hat{\mathbf{r}}(s)\big)\, ds,$$

where $\hat{\mathbf{r}}(s)$ is the arc length parametrization of the function $\mathbf{r}$, and $s(t_0) = s_0$, $s(t_1) = s_1$ are the arc lengths at the points t_0, t_1, respectively.



Line integrals in space.

Remarks:

- A line integral is an integral of a function along a curved path.
- Why is the function $\mathbf{r}$ parametrized with its arc length?
  (1) Because in this way the line integral is independent of the original parametrization of the curve. Given two different parametrizations of the curve, we have switch them to the unique arc length parametrization and compute the integral above.
  (2) Because this is the appropriate generalization of the integral of a function $F : \mathbb{R} \to \mathbb{R}$.

  Recall: $\displaystyle\int_a^b F(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n} F(x_i^*)\Delta x_i$, where $\Delta x_i = x_{i+1} - x_i$ is the distance from x_{i+1} to x_1.

  This Δx_i generalizes to Δs_i on a curved path. This is why the arc length parametrization is needed in the line integral.

Line integrals in space.

Theorem (Arbitrary parametrization.)

The line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along a differentiable curve $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C f\, ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt,$$

where t is any parametrization of the vector-valued function $\mathbf{r}$.

Proof: The integration by substitution formula says

$$\int_{s_0}^{s_1} f\big(\hat{\mathbf{r}}(s)\big)\, ds = \int_{t_0}^{t_1} f\big[\hat{\mathbf{r}}(s(t))\big]\, s'(t)\, dt, \qquad \begin{aligned} s_0 &= s(t_0), \\ s_1 &= s(t_1). \end{aligned}$$

The arc length function is $s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau$, then $s'(t) = |\mathbf{r}'(t)|$. Noticing that $\hat{\mathbf{r}}(s(t)) = \mathbf{r}(t)$, then

$$\int_C f\, ds = \int_{s_0}^{s_1} f\big(\hat{\mathbf{r}}(s)\big)\, ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt. \qquad \square$$

Line integrals in space.

Example

Evaluate the line integral of the function $f(x, y, z) = xy + y + z$ along the curve $\mathbf{r}(t) = \langle 2t, t, 2 - 2t\rangle$ in the interval $t \in [0, 1]$.

Solution: Recall: $\displaystyle\int_C f\, ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$

The derivative vector is $\mathbf{r}'(t) = \langle 2, 1, -2\rangle$, therefore its magnitude is $|\mathbf{r}'(t)| = \sqrt{4 + 1 + 4} = 3$. The values of f along the curve are

$$f(\mathbf{r}(t)) = (2t)t + t + (2 - 2t) \quad \Rightarrow \quad f(\mathbf{r}(t)) = 2t^2 - t + 2.$$

$$\int_C f\, ds = \int_0^1 (2t^2 - t + 2)\, 3\, dt = 3\Big[\Big(2\frac{t^3}{3} - \frac{t^2}{2} + 2t\Big)\Big|_0^1\Big].$$

$$\int_C f\, ds = 3\Big(\frac{2}{3} - \frac{1}{2} + 2\Big) = 2 - \frac{3}{2} + 6 \quad \Rightarrow \quad \int_C f\, ds = \frac{13}{2}. \ \triangleleft$$

Line integrals in space.

Example

Evaluate the line integral of the function $f(x, y, z) = \sqrt{x^2 + z^2}$ along the curve $\mathbf{r}(t) = \langle 0, a\cos(t), a\sin(t)\rangle$, in $t \in [0, \pi/2]$.

Solution: Recall: $\displaystyle\int_C f\, ds = \int_{t_0}^{t_1} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\, dt.$

The derivative vector is $\mathbf{r}'(t) = \langle 0, -a\sin(t), a\cos(t)\rangle$, therefore its magnitude is $|\mathbf{r}'(t)| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = |a|$. The values of f along the curve are

$$f(\mathbf{r}(t)) = \sqrt{0 + a^2\sin^2(t)} \quad \Rightarrow \quad f(\mathbf{r}(t)) = |a||\sin(t)|.$$

$$\int_C f\, ds = \int_0^{\pi/2} |a|\sin(t)\,|a|\, dt = a^2\Big(-\cos(t)\Big|_0^{\pi/2}\Big).$$

$$\int_C f\, ds = a^2. \qquad \triangleleft$$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- **The addition of line integrals.**
- Mass and center of mass of wires.

The addition of line integrals.

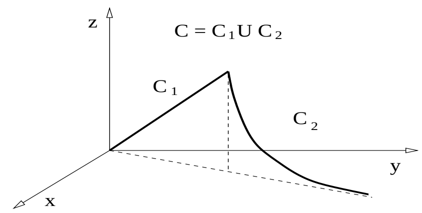
Theorem

If a curve $C \subset D$ in space is the union of the differentiable curves $C_1, \cdots, C_n$, then the line integral of a continuous function $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ along C satisfies

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds.$$

Remark:

This result is useful to compute line integral along piecewise differentiable curves.

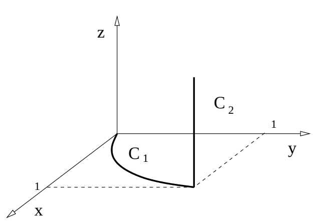


The addition of line integrals.

Example

Evaluate the line integral of $f(x,y,z) = x + \sqrt{y} - z^2$ along the path $C = C_1 \cup C_2$, where C_1 is the image of $\mathbf{r}_1(t) = \langle t, t^2, 0\rangle$ for $t \in [0,1]$, and C_2 is the image of $\mathbf{r}_2(t) = \langle 1, 1, t\rangle$ for $t \in [0,1]$.

Solution:



$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds.$$

$$\mathbf{r}_1'(t) = \langle 1, 2t, 0\rangle \Rightarrow |\mathbf{r}_1'(t)| = \sqrt{1+4t^2}.$$

$$f(\mathbf{r}_1(t)) = t + t = 2t.$$

$$\int_{C_1} f\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt, \quad u = 1+4t^2,\ du = 8t\,dt.$$

$$\int_{C_1} f\,ds = \frac{1}{4}\int_1^5 u^{1/2}\,du = \frac{1}{4}\,\frac{2}{3}\left(u^{3/2}\Big|_1^5\right) \Rightarrow \int_{C_1} f\,ds = \frac{1}{6}(5\sqrt{5}-1).$$

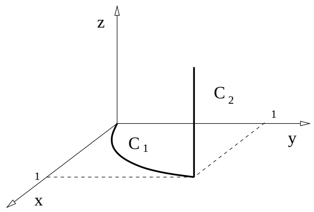
The addition of line integrals.

Example

Evaluate the line integral of $f(x,y,z) = x + \sqrt{y} - z^2$ along the path $C = C_1 \cup C_2$, where C_1 is the image of $\mathbf{r}_1(t) = \langle t, t^2, 0\rangle$ for $t \in [0,1]$, and C_2 is the image of $\mathbf{r}_2(t) = \langle 1, 1, t\rangle$ for $t \in [0,1]$.

Solution:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds.$$



$$\mathbf{r}_2'(t) = \langle 0,0,1\rangle \Rightarrow |\mathbf{r}_2'(t)| = 1.$$

$$f(\mathbf{r}_2(t)) = 1 + 1 - t^2 = 2 - t^2.$$

$$\int_{C_2} f\,ds = \int_0^1 (2 - t^2)\,dt = 2\Big(t\Big|_0^1\Big) - \Big(\frac{t^3}{3}\Big|_0^1\Big) = 2 - \frac{1}{3} = \frac{5}{3}.$$

$$\int_C f\,ds = \frac{1}{6}(5\sqrt{5} - 1) + \frac{5}{3} \quad \Rightarrow \quad \int_{C_1} f\,ds = \frac{1}{6}(5\sqrt{5} + 9). \quad \triangleleft$$

Integrals along a curve in space. (Sect. 16.1)

- Line integrals in space.
- The addition of line integrals.
- **Mass and center of mass of wires.**

Mass and center of mass of wires.

Remark:
The total mass, the center of mass, and the moments of inertia of wires with arbitrary shapes in space, given by a curve C and having a density function ρ, can be computed using line integrals.

- $M = \int_C \rho \, ds$;
- $\overline{x} = \dfrac{1}{M} \displaystyle\int_C x\rho \, ds, \;\; \overline{y} = \dfrac{1}{M} \displaystyle\int_C y\rho \, ds, \;\; \overline{z} = \dfrac{1}{M} \displaystyle\int_C z\rho \, ds$;
- $I_x = \dfrac{1}{M} \displaystyle\int_C (y^2 + z^2)\rho \, ds$,
- $I_y = \dfrac{1}{M} \displaystyle\int_C (x^2 + z^2)\rho \, ds$,
- $I_z = \dfrac{1}{M} \displaystyle\int_C (x^2 + y^2)\rho \, ds$.

Mass and center of mass of wires.

Example
Find the moments of inertia of a wheel of radius R and density ρ_0.

Solution: We place the wheel at the center of the $z = 0$ plane. The curve for the wheel is $\mathbf{r}(t) = \langle R\cos(t), R\sin(t), 0\rangle$, $t \in [0, 2\pi]$.
Therefore, $\mathbf{r}'(t) = \langle -R\sin(t), R\cos(t), 0\rangle$, hence $|\mathbf{r}'(t)| = R$.
Recall: $I_x = \int_C (y^2 + z^2)\rho_0 \, ds$, $\; I_z = \int_C (x^2 + y^2)\rho_0 \, ds$.

$$I_x = \int_0^{2\pi} R^2 \sin^2(t)\rho_0 R \, dt = R^3 \rho_0 \int_0^{2\pi} \frac{1}{2}\big[1 - \cos(2t)\big] \, dt$$

$$I_x = R^3 \rho_0 \Big[\pi - \frac{1}{4}\Big(\sin(2t)\Big|_0^{2\pi}\Big)\Big] \quad \Rightarrow \quad I_x = \pi R^3 \rho_0.$$

By symmetry, $I_x = I_y$. Finally,

$$I_z = \int_0^{2\pi} R^2 \rho_0 R \, dt \quad \Rightarrow \quad I_z = 2\pi R^3 \rho_0.$$

$\triangleleft$

Integrals of vector fields. (Sect. 16.2)

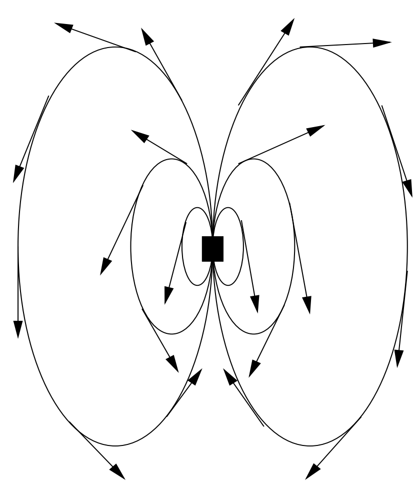
- Vector fields on a plane and in space.
  - The gradient field of a scalar-valued function.
- The line integral of a vector field along a curve.
  - Work done by a force on a particle.
  - The flow of a fluid along a curve.
- The flux across a plane curve.

Vector fields on a plane and in space.

Definition
A *vector field* on a plane or in space is a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, respectively.

Examples from physics:

- Electric and magnetic fields.
- The gravitational field of the Earth.
- The velocity field in a fluid or gas.
- The variation of temperature in a room. (Gradient field.)



Magnetic field of a small magnet

Integrals of vector fields. (Sect. 16.2)

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The gradient field of a scalar-valued function.

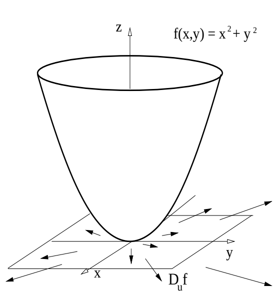
Remark:

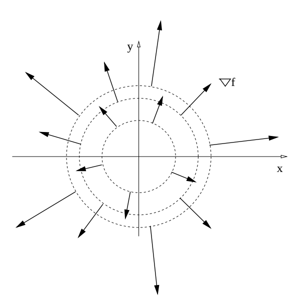
- Given a scalar-valued function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3$, its gradient vector, $\nabla f = \langle \partial_x f, \partial_y f \rangle$ or $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$, respectively, is a vector field in a plane or in space.

Example

Find and sketch a graph of the gradient field of the function $f(x, y) = x^2 + y^2$.

Solution: We know the graph of f is a paraboloid. The gradient field is $\nabla f = \langle 2x, 2y \rangle$.





Integrals of vector fields. (Sect. 16.2)

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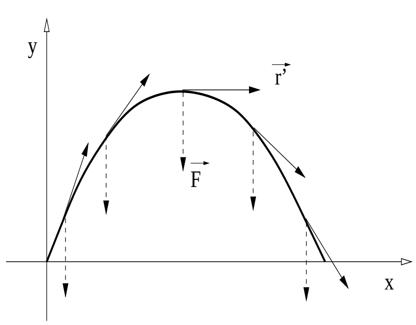
The line integral of a vector field along a curve.

Definition

The *line integral* of a vector-valued function $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, along the curve associated with the function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt$$

Example



Remark: An equivalent expression is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt,$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \hat{\mathbf{F}} \cdot \hat{\mathbf{u}}\, ds,$$

where $\hat{\mathbf{u}} = \dfrac{\mathbf{r}'(t(s))}{|\mathbf{r}'(t(s))|}$, and $\hat{\mathbf{F}} = \mathbf{F}(t(s))$.

Integrals of vector fields. (Sect. 16.2)

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Work done by a force on a particle.

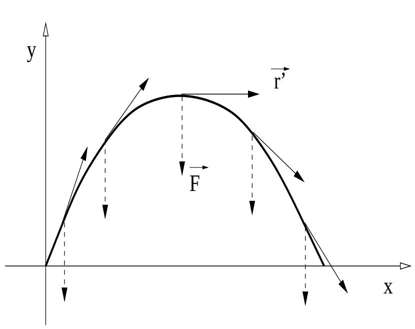
Definition

In the case that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, represents a force acting on a particle with position function $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$, then the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

is called the *work* done by the force on the particle.

Example



A projectile of mass m moving on the surface of Earth.

- The movement takes place on a plane, and $\mathbf{F} = \langle 0, -mg \rangle$.
- $W \leqslant 0$ in the first half of the trajectory, and $W \geqslant 0$ on the second half.

Work done by a force on a particle.

Example

Find the work done by the force $\mathbf{F}(x,y,z) = \langle (3x^2 - 3x), 3z, 1\rangle$ on a particle moving along the curve with $\mathbf{r}(t) = \langle t, t^2, t^4\rangle$, $t \in [0,1]$.

Solution:

First: Evaluate $\mathbf{F}$ along $\mathbf{r}$. This is: $\mathbf{F}(t) = \langle (3t^2 - 3t), 3t^4, 1\rangle$.

Second: Compute $\mathbf{r}'(t)$. This is: $\mathbf{r}'(t) = \langle 1, 2t, 4t^3\rangle$.

Third: Integrate the dot product $\mathbf{F}(t) \cdot \mathbf{r}'(t)$.

$$W = \int_0^1 \big[(3t^2 - 3t) + (6t^5) + (4t^3)\big]\, dt$$

$$= \Big(t^3 - \frac{3}{2}t^2 + t^6 + t^4\Big)\Big|_0^1 = 1 - \frac{3}{2} + 1 + 1.$$

So, $W = 3 - \dfrac{3}{2}$. We conclude: The work done is $W = \dfrac{3}{2}$. $\triangleleft$

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The flow of a fluid along a curve.

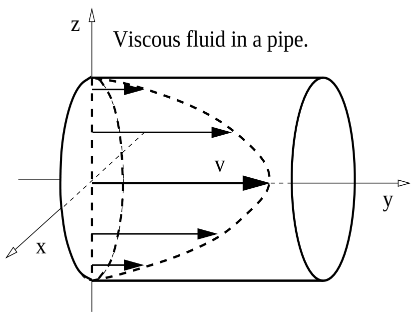
Definition

In the case that the vector field $\mathbf{v} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2, 3$, is the velocity field of a flow and $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to D \subset \mathbb{R}^3$ is any smooth curve, then the line integral

$$F = \int_C \mathbf{v} \cdot d\mathbf{r},$$

is called a *flow integral*. If the curve is a closed loop, the flow integral is called the *circulation* of the fluid around the loop.

Example



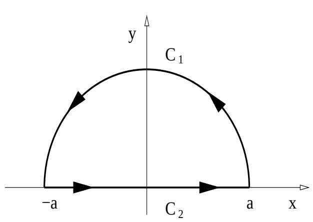
- The flow of a viscous fluid in a pipe is maximal along a line through the center of the pipe.
- The flow vanishes on any curve perpendicular to the section of the pipe.

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x \rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t) \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0 \rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v} \cdot d\mathbf{r}_2$.



The first term is given by:

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} \mathbf{v}(t) \cdot \mathbf{r}_1'(t)\, dt.$$

$$\mathbf{v}(t) = \langle -a\sin(t), a\cos(t) \rangle,$$

$$\mathbf{r}_1'(t) = \langle -a\sin(t), a\cos(t) \rangle.$$

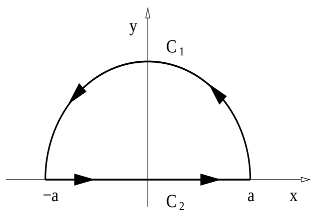
$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_0^{\pi} a^2 \left[\sin^2(t) + \cos^2(t)\right] dt \quad \Rightarrow \quad \int_{C_1} \mathbf{v} \cdot d\mathbf{r}_1 = \pi a^2.$$

The flow of a fluid along a curve.

Example

Find the circulation of a fluid with velocity field $\mathbf{v} = \langle -y, x\rangle$ along the closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t)\rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0\rangle$ for $t \in [-a, a]$.

Solution: The circulation is: $F = \int_{C_1} \mathbf{v}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{v}\cdot d\mathbf{r}_2$.



The second term is given by:

$$\int_{C_2} \mathbf{v}\cdot d\mathbf{r}_2 = \int_{-a}^{a} \mathbf{v}(t)\cdot \mathbf{r}_2'(t)\,dt,$$

$$\mathbf{v}(t) = \langle 0, t\rangle, \quad \mathbf{r}_2'(t) = \langle 1, 0\rangle.$$

$$\mathbf{v}(t)\cdot \mathbf{r}_2'(t) = 0 \quad \Rightarrow \quad \int_{C_2} \mathbf{v}\cdot d\mathbf{r}_2 = 0.$$

Since $\int_{C_1} \mathbf{v}\cdot d\mathbf{r}_1 = \pi a^2$, we conclude: $F = \pi a^2$. $\lhd$

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- **The flux across a plane curve.**

The flux across a plane curve.

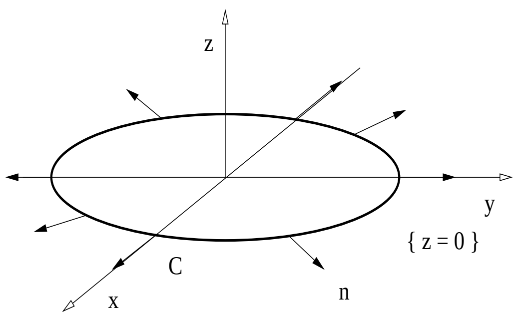
Definition

The *flux* of a vector field $\mathbf{F} : \{z = 0\} \subset \mathbb{R}^3 \to \{z = 0\} \subset \mathbb{R}^3$ along a closed plane loop $\mathbf{r} : [t_0, t_1] \subset \mathbb{R} \to \{z = 0\} \subset \mathbb{R}^3$ is given by

$$\mathbb{F} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the unit outer normal vector to the curve inside the plane $\{z = 0\}$.

Example



Remarks:

- $\mathbf{F}$ is defined on $\{z = 0\}$.
- The loop C lies on $\{z = 0\}$.
- Simple formula for $\mathbf{n}$?

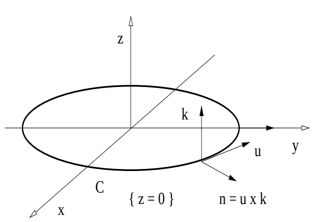
The flux across a plane curve.

Theorem (Counterclockwise loops.)

The flux of a vector field $\mathbf{F} = \langle F_x(x, y), F_y(x, y), 0\rangle$ along a closed, counterclockwise plane loop $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$ for $t \in [t_0, t_1]$ is given by

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{t_0}^{t_1} \big[F_x\, y'(t) - F_y\, x'(t)\big]\, dt.$$

Proof:



Remarks: Since C is counterclockwise traversed, $\mathbf{n} = \mathbf{u} \times \mathbf{k}$, where $\mathbf{u} = \mathbf{r}'/|\mathbf{r}'|$.

$$\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|}\langle x'(t), y'(t), 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle.$$

$$\mathbf{n} = \frac{1}{|\mathbf{r}'|}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \Rightarrow \quad \mathbf{n} = \frac{1}{|\mathbf{r}'|}\,\langle y'(t), -x'(t), 0\rangle.$$

The flux across a plane curve.

Theorem (Counterclockwise loops.)

The flux of a vector field $\mathbf{F} = \langle F_x(x,y), F_y(x,y), 0\rangle$ *along a closed, counterclockwise plane loop* $\mathbf{r}(t) = \langle x(t), y(t), 0\rangle$ *for* $t \in [t_0, t_1]$ *is given by*

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[F_x\,y'(t) - F_y\,x'(t)\big]\,dt.$$

Proof: Recall: $\mathbf{n} = \dfrac{1}{|\mathbf{r}'|}\,\langle y'(t), -x'(t), 0\rangle.$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \langle F_x, F_y, 0\rangle\cdot\langle y'(t), -x'(t), 0\rangle\,\frac{1}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|\,dt$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{t_0}^{t_1} \big[F_x\,y'(t) - F_y\,x'(t)\big]\,dt. \qquad \square$$

The flux across a plane curve.

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0\rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0\rangle$ for $t \in [0,\pi]$, and $\mathbf{r}_2 = \langle t, 0, 0\rangle$ for $t \in [-a, a]$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_{C_1} \mathbf{F}_1\cdot\mathbf{n}_1\,ds + \int_{C_2} \mathbf{F}_2\cdot\mathbf{n}_2\,ds$

Along C_1 we have: $\mathbf{F}_1(t) = \langle -a\sin(t), a\cos(t), 0\rangle$ and

$$x'(t) = -a\sin(t), \quad y'(t) = a\cos(t).$$

Therefore,

$$F_{1x}(t)\,y'(t) - F_{1y}(t)\,x'(t) = -a^2\sin(t)\cos(t) + a^2\sin(t)\cos(t) = 0.$$

Hence: $\displaystyle\int_{C_1} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$

The flux across a plane curve.

Example

Find the flux of a field $\mathbf{F} = \langle -y, x, 0 \rangle$ across the plane closed loop given by $\mathbf{r}_1 = \langle a\cos(t), a\sin(t), 0 \rangle$ for $t \in [0, \pi]$, and $\mathbf{r}_2 = \langle t, 0, 0 \rangle$ for $t \in [-a, a]$.

Solution: Recall: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_{C_1} \mathbf{F}_1 \cdot \mathbf{n}_1\, ds + \int_{C_2} \mathbf{F}_2 \cdot \mathbf{n}_2\, ds$

Along C_2 we have: $\mathbf{F}_2(t) = \langle 0, t, 0 \rangle$ and $x'(t) = 1$, $y'(t) = 0$. So,

$$F_{2x}(t)\, y'(t) - F_{2y}(t)\, x'(t) = 0 - t \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = \int_{-a}^{a} -t\, dt,$$

$$\int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = -\left(\frac{t^2}{2} \Big|_{-a}^{a} \right) \quad \Rightarrow \quad \int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds = 0.$$

We conclude: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0.$ $\triangleleft$