Math 234, Practice Test #2

Show your work in all the problems.

1. In what directions is the derivative of

\[ f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \]

at \( P = (1, 1) \) equal to zero?

2. Find an equation for the level surface of the function through the given point \( P \).

(a)

\[ f(x, y, z) = z - x^2 - y^2, \quad P = (3, -1, 1) \]

(b)

\[ f(x, y, z) = \int_x^y \frac{dt}{\sqrt{1 - t^2}} + \int_{\sqrt{x}}^{\sqrt{y}} \frac{dt}{t\sqrt{t^2 - 1}}, \quad P = (-1, 1/2, 1) \]

3. Compute the limits of the following expressions if they exist. If you think they don’t, consider different paths of approach to show that they do not.

(a)

\[ \frac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}, \quad (x, y) \to (4, 3), \quad x \neq y + 1 \]

(b)

\[ \frac{x^2 + y^2}{xy}, \quad (x, y) \to (0, 0), \quad xy \neq 0 \]

4. Compute all second order partial derivatives of the function

\[ f(x, y) = x \sin y + y \sin x + xy \]

5. Find \( \frac{dw}{dt} \) if \( w = \sin(xy + \pi), \ x = e^t \) and \( y = \ln(t + 1) \). Then evaluate at \( t = 0 \).
Solutions

1. We first compute the gradient vector of $f$:

$$\nabla f = \left( \frac{4xy^2}{(x^2 + y^2)^2}, -\frac{4x^2y}{(x^2 + y^2)^2} \right)$$

Evaluating at $(1, 1)$ yields

$$\nabla f(1, 1) = (1, -1).$$

The directions $u$ in which the directional derivative $(D_u f)(1, 1)$ is zero are the unit vectors orthogonal to $\nabla f(1, 1) = (1, -1)$, i.e. $u = (u_1, u_2)$ has to satisfy

$$\nabla f(1, 1) \cdot u = (1, -1) \cdot (u_1, u_2) = u_1 - u_2 = 0$$

and $u_1^2 + u_2^2 = 1$. We get

$$u = \frac{1}{\sqrt{2}} (1, 1) \text{ and } -u = -\frac{1}{\sqrt{2}} (1, 1).$$

2. (a) The level surfaces are given by the equations

$$c = z - x^2 - y^2$$

where $c$ is a constant. In order to single out the one which passes through the point $P$ we need to insert the coordinates of $P$ and compute the constant $c$. Hence

$$c = 1 - 3^2 - (-1)^2 = -9$$

so that the desired equation is $-9 = z - x^2 - y^2$.

(b) We compute the integrals first

$$\int_x^y \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1}(y) - \sin^{-1}(x)$$

and

$$\int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2 - 1}} = \sec^{-1}|z| - \sec^{-1}(\sqrt{2}) = \sec^{-1}|z| - \frac{\pi}{4}$$
The latter follows from the fact that sec$^{-1}(\sqrt{2})$ is the 'angle' $\theta$ where $\cos \theta = 1/\sqrt{2}$ which is $\pi/4$. We insert now $x = -1$, $y = 1/2$ and $z = 1$:

\[
c = f(-1, 1/2, 1) = \sin^{-1}(1/2) - \sin^{-1}(-1) + \sec^{-1} |1| - \frac{\pi}{4} = \frac{\pi}{6} + \frac{\pi}{2} + 0 - \frac{\pi}{4} = \frac{5\pi}{12}
\]

The equation of the level surface is then

\[
\frac{5\pi}{12} = \sin^{-1}(y) - \sin^{-1}(x) + \sec^{-1} |z| - \frac{\pi}{4}
\]

or

\[
\frac{2\pi}{3} = \sin^{-1}(y) - \sin^{-1}(x) + \sec^{-1} |z|
\]

3. (a) We use the following formula to simplify the given expression

\[
(\sqrt{x} - \sqrt{y + 1})(\sqrt{x} + \sqrt{y + 1}) = (\sqrt{x})^2 - (\sqrt{y + 1})^2 = x - (y + 1) = x - y - 1
\]

Then

\[
\frac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1} = \frac{1}{\sqrt{x} + \sqrt{y + 1}}
\]

and the limit can simply be computed by inserting $(x, y) = (4, 3)$ which yields $1/4$.

(b) No further simplification as in problem (a) is possible here. We look at the level curves

\[
c = \frac{x^2 + y^2}{xy} \text{ or } x^2 + y^2 - cxy = 0
\]

Solving this for $y$ (quadratic equation) we get

\[
y = \frac{1}{2}(c \pm \sqrt{c^2 - 4}) x = kx.
\]
Note that all these curves pass through the origin, so there is no limit for \((x, y) \to (0, 0)\). In order to confirm, we insert \(y = kx\) and we get
\[
\frac{x^2 + y^2}{xy} = \frac{x^2 + k^2x^2}{kx^2} = \frac{1 + k^2}{k}
\]
Taking the limit \(x \to 0\) still yields \((1 + k^2)/k\), i.e. the value depends on the direction in which we approach the origin.

4. We have

\[
f(x, y) = x \sin y + y \sin x + xy
\]
The first order derivatives are given by

\[
f_x = \sin y + y \cos x + y, \quad f_y = x \cos y + \sin x + x
\]

Then

\[
f_{xx} = -y \sin x, \quad f_{yy} = -x \sin y, \quad f_{xy} = f_{yx} = \cos y + \cos x + 1
\]

5. We compute

\[
\frac{\partial w}{\partial x} = y \cos(xy + \pi), \quad \frac{\partial w}{\partial y} = x \cos(xy + \pi)
\]
and

\[
x'(t) = e^t, \quad y'(t) = \frac{1}{t + 1}.
\]

Then

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial y} y'(t)
\]
\[
= e^t \ln(t + 1) \cos(e^t \ln(t + 1) + \pi) + \frac{e^t}{t + 1} \cos(e^t \ln(t + 1) + \pi)
\]
\[
= \left(\ln(t + 1) + \frac{1}{t + 1}\right) e^t \cos(e^t \ln(t + 1) + \pi)
\]
Evaluating at \(t = 0\) yields

\[
\frac{dw}{dt} \bigg|_{t=0} = \cos(\pi) = -1
\]