Math 234, Practice Test #1

Show your work in all the problems.

1. Find parametric equations for the line in which the planes \( x+2y+z = 1 \) and \( x - y + 2z = -8 \) intersect.

2. Compute the distance from the point \((2, 2, 3)\) to the plane through the points \(A = (0, 0, 0), B = (2, 0, -1)\) and \(C = (2, -1, 0)\).

3. Compute the area of the parallelogram with three of its vertices given by
   \[ A = (2, -2, 1), \quad B = (3, -1, 2) \quad \text{and} \quad C = (3, -1, 1) \]

4. \((Cancellation\ in\ a\ dot\ product\ ?)\) Let \(u, v, w\) be three vectors with \(u \neq 0\). Is it true that \(u \cdot v = u \cdot w\) implies \(v = w\)? If you think it is true explain why, otherwise provide a counterexample.

5. Sketch the surface given by the equation \(z = 1 - x^2\).

6. Describe the given sets with a single equation or a pair of equations:
   The circle of radius 1 centered at \((-3, 4, 1)\) and lying in a plane parallel to the
   (a) xy-plane
   (b) yz-plane
   (c) xz-plane
Solutions

1. Normal vectors of the two planes are given by

\[ \mathbf{n}_1 = (1, 2, 1) \text{ and } \mathbf{n}_2 = (1, -1, 2) \]

respectively. The line of intersection is perpendicular to both \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \). The following vector is then parallel to the line of intersection:

\[
\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2 & 1 \\
1 & -1 & 2
\end{vmatrix}
\]

\[ = \begin{vmatrix}
2 & 1 \\
-1 & 2
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
1 & 1 \\
1 & 2
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
1 & 2 \\
1 & -1
\end{vmatrix} \mathbf{k}
\]

\[ = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}
\]

\[ = (5, -1, -3) \]

We also need to find a point which lies on the line of intersection. We insert \( x = 1 - 2y - z \) (which is derived from the first equation) into the second equation, and we get

\[ -8 = x - y + 2z = 1 - 2y - z - y + 2z = 1 - 3y + z \]

as well as \( z = -9 + 3y \) and \( x = 1 - 2y - z = 10 - 5y \). This means we can pick any value we want for \( y \) and compute \( x, z \) using the previous formulas. For \( y = 0 \) we get \( x = 10 \) and \( z = -9 \) so that the point \((10, 0, -9)\) lies on the line of intersection. Parametric equations of the line are then given by

\[ x = 10 + 5t, \quad y = -t, \quad z = -9 - 3t \]

Comment: There are many possible solutions which are all correct (and for which you would get full credit). For example, the vector \((-5, 1, 3)\) is also parallel to the line of intersection (do you know why?). Later on, instead of choosing \( y = 0 \) to find a point on the line we could have chosen something else, for example \( y = 1 \). We would have obtained \( x = 10 - 5y = 5 \) and \( z = -9 + 3y = -6 \), so that we use the point \((5, 1, -6)\) instead. Remember that there are many points on a line,
and there is no reason to prefer one over the other. The parametric equations would then look as follows

\[ x = 5 - 5t, \ y = 1 + t, \ z = -6 + 3t. \]

Although these equations are different they describe the same line, and they are therefore also a valid solution to the problem.

2. We need to find a normal vector \( \mathbf{n} \) to the plane, i.e. a vector perpendicular to both \( \vec{AB} = (2, 0, -1) \) and \( \vec{AC} = (2, -1, 0) \). We get a such a vector by taking the cross product

\[
\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}
\]

If \( P \) is any point in the plane then the distance \( d \) between \( S = (2, 2, 3) \) and the plane is given by

\[
d = \frac{|\vec{PS} \cdot \mathbf{n}|}{|\mathbf{n}|}
\]

The easiest pick for \( P \) is probably \( (0, 0, 0) \), so that \( \vec{PS} = (2, 2, 3) \). We get

\[
\vec{PS} \cdot \mathbf{n} = (2, 2, 3) \cdot (-1, -2, -2) = -2 - 4 - 6 = -12
\]

and

\[
|\mathbf{n}| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3
\]

so that

\[
d = \frac{|-12|}{3} = 4.
\]

3. The area of the parallelogram is given by

\[
|\vec{AB} \times \vec{AC}|
\]

We have

\[
\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} = (-1, 1, 0)
\]

and

\[
|\vec{AB} \times \vec{AC}| = \sqrt{2}.
\]
4. The statement is false. Pick two different vectors $v$ and $w$, for example $v = (1, 0, 0)$ and $w = (0, 1, 0)$. Then choose $u$ perpendicular to both $v$ and $w$, for example $u = (0, 0, 1)$ would do. Then

$$u \cdot v = u \cdot w = 0$$

but $v \neq w$.

5. The surface is a cylinder over the parabola $z = 1 - x^2$ in the $xz$-plane:

6. (a) $(x + 3)^2 + (y - 4)^2 = 1$, $z = 1$
   (b) $(y - 4)^2 + (z - 1)^2 = 1$, $x = -3$
   (c) $(x + 3)^2 + (z - 1)^4 = 1$, $y = 4$