1. (26 points) Consider the integral \( \int \int_{D} f(x, y) \, dA = \int_{0}^{3} \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2\sqrt{1-\frac{x^2}{3^2}}} f(x, y) \, dy \, dx \).

(a) (8 points) Sketch the region of integration.

(b) (8 points) Switch the order of integration in the above integral.

(c) (10 points) Compute the integral \( \int \int_{D} f(x, y) \, dA \) for the case \( f(x, y) = xy \).

Solution:

(a) 
- The limits in \( x \): \( x \in [0, 3] \).
- The limits in \( y \):
  - Upper limit: \( y = 2\left(1 - \frac{x}{3}\right) \).
  - Lower limit: \( y = -2\sqrt{1 - \frac{x^2}{3^2}} \); so a part of the ellipse \( \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1 \).

(b) If we integrate first in \( x \), we need to split the integral at \( y = 0 \). In the interval \( y \in [-2, 0] \), the lower limit in \( x \) is \( 0 \leq x \). The upper limit comes from \( \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1 \), that is, \( x = +3\sqrt{1 - \frac{y^2}{2^2}} \).

In the interval \( y \in [0, 2] \), the lower limit in \( x \) is again \( 0 \leq x \). The upper limit comes from \( y = 2\left(1 - \frac{x}{3}\right) \), that is, \( x = 3\left(1 - \frac{y}{2}\right) \).

We then conclude:

\[
\int \int_{D} f(x, y) \, dA = \int_{-2}^{0} \int_{0}^{3\sqrt{1-\frac{y^2}{2^2}}} f(x, y) \, dx \, dy + \int_{0}^{2} \int_{0}^{3(1-\frac{y}{2})} f(x, y) \, dx \, dy.
\]
(c) This is a straightforward, albeit long, calculation. We can use any of the two order of integration we have for $I$. We choose the shorter one:

$$I = \int_0^3 \int_{-2\sqrt{1-x^2/3}}^{2(1-x^2/3)} xy \, dy \, dx = \int_0^3 x \left( \frac{y^2}{2} \right)_{-2\sqrt{1-x^2/3}}^{2(1-x^2/3)} \, dx,$$

$$I = \frac{1}{2} \int_0^3 x \left[ 4 \left( 1 - \frac{x}{3} \right)^2 - 4 \left( 1 - \frac{x^2}{3^2} \right) \right] \, dx = 2 \int_0^3 x \left( 1 + \frac{x^2}{3^2} - \frac{x}{3} + \frac{x^2}{3} \right) \, dx,$$

$$I = 2 \int_0^3 x \left( \frac{2x^2}{3^2} - \frac{2x}{3} \right) \, dx = \frac{4}{3^2} \int_0^3 \left( x^3 - 3x^2 \right) \, dx,$$

$$I = \frac{4}{3^2} \left( \frac{x^4}{4} - x^3 \right) \bigg|_0^3 = \frac{4}{3^2} \left( \frac{3^4}{4} - 3^3 \right) = 4 \left( \frac{3^2}{4} - 3 \right),$$

$$I = (9 - 12) = -5, \quad \Rightarrow \quad I = -5.$$
2. (20 points) Find the component $x$ of the centroid vector in Cartesian coordinates in the plane of the region $R = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \quad y \geq 0, \quad x^2 + y^2 \leq 2^2\}$.

**Solution:**

If $A$ denotes the area of the region, then the centroid vector $\vec{r} = (\bar{x}, \bar{y})$ is given by:

$$\bar{x} = \frac{1}{A} \iint_R x \, dx \, dy, \quad \bar{y} = \frac{1}{A} \iint_R y \, dx \, dy.$$ 

From the figure we see that the region is a quarter of a disk, hence $A = \frac{1}{4} \pi 2^2$, that is, $[A = \pi]$. Then, $\bar{x}$ is given by:

One way is:

$$\bar{x} = \frac{1}{\pi} \int_0^2 \int_0^{\sqrt{4-x^2}} x \, dy \, dx,$$

substitution: $u = 4 - x^2$, $du = -2x \, dx$,

$$\bar{x} = \frac{1}{\pi} \int_0^2 \int_0^{\sqrt{4-u}} \frac{u^{1/2}}{2} \, du,$$

$$= \frac{1}{2\pi} \int_0^2 u^{1/2} \, du,$$

$$= \frac{1}{2\pi} \left[ \frac{2}{3} u^{3/2} \right]_0^4,$$

$$= \frac{1}{3\pi} 8 \quad \Rightarrow \quad \boxed{\bar{x} = \frac{8}{3\pi}}.$$  

Another ways is:

$$\bar{x} = \frac{1}{\pi} \int_0^2 \int_0^{\sqrt{4-y^2}} x \, dx \, dy,$$

$$= \frac{1}{\pi} \int_0^2 \left( \frac{x^2}{2} \right)_0^{\sqrt{4-y^2}} \, dy,$$

$$= \frac{1}{2\pi} \int_0^2 (4 - y^2) \, dy,$$

$$= \frac{1}{2\pi} \left( 4y \bigg|_0^2 - \left( \frac{y^3}{3} \right)_0^2 \right),$$

$$= \frac{1}{2\pi} \left( 8 - \frac{8}{3} \right),$$

$$= \frac{4}{\pi} \frac{2}{3} \quad \Rightarrow \quad \boxed{\bar{x} = \frac{8}{3\pi}}.$$  

Anyway it is correct.
3. (16 points) Transform to polar coordinates and then evaluate the integral

\[ I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left( x^2 + y^2 \right)^{3/2} \, dx \, dy. \]

**Solution:**

It is helpful to sketch the integration region:

- Limits in \( y \): \( y \in [-1, 1] \).
- Limits in \( x \):
  - Lower limit \( x = 0 \), upper limit the curve \( x = \sqrt{1-y^2} \), that is, the circle \( x^2 + y^2 = 1 \).

Therefore, the integral \( I \) in polar coordinates is the following

\[
I = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \left( r^2 \right)^{3/2} (r \, dr) \, d\theta,
\]

\[
= \left( \int_{-\pi/2}^{\pi/2} d\theta \right) \left( \int_{0}^{1} r^4 \, dr \right),
\]

\[
= \pi \left( \frac{r^5}{5} \right)_0^1,
\]

\[
= \frac{\pi}{5} \quad \Rightarrow \quad I = \frac{\pi}{5}.
\]
4. (18 points) Find the volume of a parallelepiped whose base is a rectangle in the \( z = 0 \) plane given by \( 0 \leq y \leq 1 \) and \( 0 \leq x \leq 2 \), while the top side lies in the plane \( x + y + z = 3 \).

\[ V = \int_0^2 \int_0^1 \int_0^{3-x-y} dz \, dy \, dx \]

\[ = \int_0^2 \int_0^1 (3 - x - y) \, dy \, dx, \]

\[ = \int_0^2 \left[ (3 - x)\left(y\bigg|_0^1\right) - \frac{1}{2}\left(y^2\bigg|_0^1\right) \right] dx, \]

\[ = \int_0^2 (3 - x - \frac{1}{2}) \, dx, \]

\[ = \int_0^2 \left( \frac{5}{2} - x \right) \, dx, \]

\[ = \left[ \frac{5}{2}x\bigg|_0^2 - \frac{1}{2}x^2\bigg|_0^2 \right], \]

\[ = 5 - 2, \]

\[ = 3 \Rightarrow V = 3. \]
5. (20 points) Consider the region of $R \subset \mathbb{R}^3$ given by

$$R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1 + x^2 + y^2\}.$$

(a) (5 points) Sketch the region $R$.

(b) (15 points) Use cylindrical coordinates to compute the volume of that region.

Solution:

(a)

- The condition $x^2 + y^2 \leq 1$ implies $r \leq 1$.
- The last condition is $0 \leq z \leq 1 + r^2$.
- No further conditions, so $\theta \in [0, 2\pi]$.

(b) The calculation is simple, once we have the appropriate integration limits.

$$V = \int_0^{2\pi} \int_0^1 \int_0^{1+r^2} dz \, r \, dr \, d\theta,$$

$$= 2\pi \int_0^1 (1 + r^2) r \, dr,$$

Substitution: $u = 1 + r^2, \quad du = 2r \, dr$,

$$V = 2\pi \int_1^2 \frac{u}{2} du,$$

$$= 2\pi \left( \frac{1}{2} \left( \frac{u^2}{2} \right) \right|_1^2,$$

$$= \pi \left( 2 - \frac{1}{2} \right) \quad \Rightarrow \quad V = \frac{3\pi}{2}. $$
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