1. (a) (15 points) Find the position $\mathbf{r}$ and velocity vector functions $\mathbf{v}$ of a particle that moves with an acceleration function $\mathbf{a}(t) = \langle 0, 0, -10 \rangle \text{ m/sec}^2$, knowing that the initial velocity and position are given by, respectively, $\mathbf{v}(0) = \langle 0, 1, 2 \rangle \text{ m/sec}$ and $\mathbf{r}(0) = \langle 0, 0, 3 \rangle \text{ m}$.

(b) (5 points) Draw an approximate picture of the graph of $\mathbf{r}(t)$ for $t \geq 0$.

**Solution:**

(a) 

$$\mathbf{a}(t) = \langle 0, 0, -10 \rangle,$$

$$\mathbf{v}(t) = \langle v_{0x}, v_{0y}, -10t + v_{0z} \rangle, \quad \mathbf{v}(0) = \langle 0, 1, 2 \rangle \implies \begin{cases} v_{0x} = 0, \\
v_{0y} = 1, \\
v_{0z} = 2. \end{cases}$$

$$\mathbf{v}(t) = \langle 0, 1, -10t + 2 \rangle.$$ 

$$\mathbf{r}(t) = \langle r_{0x}, t + r_{0y}, -5t^2 + 2t + r_{0z} \rangle, \quad \mathbf{r}(0) = \langle 0, 0, 3 \rangle \implies \begin{cases} r_{0x} = 0, \\
r_{0y} = 0, \\
r_{0z} = 3. \end{cases}$$

$$\mathbf{r}(t) = \langle 0, t, -5t^2 + 2t + 3 \rangle.$$ 

(b)
2. (a) (10 points) Find and sketch the domain of the function \( f(x, t) = \ln(3x + 2t) \).

(b) (10 points) Find all possible constants \( c \) such that the function \( f(x, t) \) above is a solution of the wave equation, \( f_{tt} - c^2 f_{xx} = 0 \).

**Solution:**

(a) The argument in the \( \ln \) function must be positive. Then, the domain is 

\[
D = \{ (x, t) \in \mathbb{R}^2 : 3x + 2t > 0 \}.
\]

(b) 

\[
egin{align*}
  f_t &= \frac{2}{3x + 2t}, \\
  f_{tt} &= -\frac{4}{(3x + 2t)^2}, \\
  f_x &= \frac{3}{3x + 2t}, \\
  f_{xx} &= -\frac{9}{(3x + 2t)^2}, \\
  0 &= f_{tt} - c^2 f_{xx} = -\frac{4}{(3x + 2t)^2} + c^2 \frac{9}{(3x + 2t)^2} = \frac{1}{(3x + 2t)^2}(-4 + 9c^2) \\
  \Rightarrow \quad 9c^2 &= 4, \quad \Rightarrow \quad c = \pm \frac{2}{3}.
\end{align*}
\]
3. (a) (10 points) Find the direction in which \( f(x, y) \) increases the most rapidly, and the directions in which \( f(x, y) \) decreases the most rapidly at \( P_0 \), and also find the value of the directional derivative of \( f(x, y) \) at \( P_0 \) along these directions, where

\[
f(x, y) = x^3 e^{-2y}, \quad \text{and} \quad P_0 = (1, 0).
\]

(b) (10 points) Find the directional derivative of \( f(x, y) \) above at the point \( P_0 \) in the direction given by \( v = (1, -1) \).

Solution:

(a) The direction in which \( f \) increases the most rapidly is given by \( \nabla f \), and the one in which decreases the most rapidly is \( -\nabla f \). So,

\[
\nabla f(x, y) = \langle 3x^2 e^{-2y}, -2x^3 e^{-2y} \rangle, \quad \Rightarrow \quad \nabla f(1, 0) = \langle 3, -2 \rangle, \quad -\nabla f(1, 0) = \langle -3, 2 \rangle.
\]

The value of the directional derivative along these directions is, respectively, \( |\nabla f(1, 0)| \) and \(-|\nabla f(1, 0)|\), where

\[
|\nabla f(1, 0)| = \sqrt{9 + 4} = \sqrt{13}.
\]

(b) A unit vector along \( (1, -1) \) is \( u = \frac{1}{\sqrt{2}} (1, -1) \), then,

\[
D_u f(1, 0) = \nabla f(1, 0) \cdot u = \langle 3, -2 \rangle \cdot \frac{1}{\sqrt{2}} (1, -1) = \frac{5}{\sqrt{2}},
\]

\[
D_u f(1, 0) = \frac{5}{\sqrt{2}}.
\]
4. (a) (10 points) Find the tangent plane approximation of $f(x, y) = x \cos(\pi y/2) - y^2 e^{-x}$ at the point $(0, 1)$.

(b) (10 points) Use the linear approximation computed above to approximate the value of $f(-0.1, 0.9)$.

**Solution:**

(a) 
\[
\begin{align*}
  f(x, y) &= x \cos(\pi y/2) - y^2 e^{-x} & f(0, 1) &= -1, \\
  f_x(x, y) &= \cos(\pi y/2) + y^2 e^{-x} & f_x(0, 1) &= \cos(\pi/2) + 1 = 1, \\
  f_y(x, y) &= -x \sin(\pi y/2) \frac{\pi}{2} - 2y e^{-x} & f_y(0, 1) &= -2,
\end{align*}
\]

Then, the linear approximation $L(x, y)$ is given by
\[
L(x, y) = (x - 0) - 2(y - 1) - 1 \quad \Rightarrow \quad L(x, y) = x - 2y + 1.
\]

(b) The linear approximation of $f(-0.1, 0.9)$ is $L(-0.1, 0.9)$, which is given by
\[
L(-0.1, 0.9) = -0.1 - 2(-0.1) - 1 = -0.1 - 1 = -1.1, \quad \Rightarrow \quad L(-0.1, 0.9) = -1.1.
\]
5. (20 points) Find every local and absolute extrema of \(f(x, y) = x^2 + 3y^2 + 2y\) on the unit disk \(x^2 + y^2 \leq 1\), and indicate which ones are the absolute extrema. In the case of the interior stationary points, decide whether they are local maximum, minimum of saddle points.

Solution:
We first compute the interior stationary points, which are \((x, y)\) solutions of

\[
\nabla f = (2x, 6y + 2) = (0, 0) \implies x = 0, \quad y = -\frac{1}{3}.
\]

The point \((0, -1/3)\) belongs to the disk \(x^2 + y^2 \leq 1\) so we have to decide whether it is a local maximum, minimum or saddle point:

\[
f_{xx} = 2, \quad f_{yy} = 6, \quad f_{xy} = 0,
\]

\[
D = f_{xx}f_{yy} - (f_{xy})^2 = 12 > 0, \quad f_{xx} > 0 \implies \left(0, -\frac{1}{3}\right) \text{ is a local minimum}.
\]

This point is also a candidate for absolute minimum, so we record the value of \(f\),

\[
\left(0, -\frac{1}{3}\right) \implies f\left(0, -\frac{1}{3}\right) = 0 + \frac{3}{9} - \frac{2}{3} = -\frac{1}{3}.
\]

We now look for extreme point on the boundary \(x^2 + y^2 = 1\). We evaluate \(f(x, y)\) along the boundary. From the equation \(x^2 + y^2 = 1\) we compute \(x = \pm \sqrt{1 - y^2}\). This function is differentiable for \(y \in (-1, 1)\), but is not differentiable at \(y = \pm 1\). Since we need to use the chain rule to find the extrema of \(g(y) = f(x(y), y)\) and the chain rule does not hold at \(y = \pm 1\), we need to consider these points, \((0, \pm 1)\) separately:

\[
(0, 1) \implies f(0, 1) = 5, \quad (0, -1) \implies f(0, -1) = 1.
\]

Now we find local extrema on \(g(y) = f(x(y), y)\) in the interval \(y \in (-1, 1)\). The function \(g\) is given by

\[
g(y) = (1 - y^2) + 3y^2 + 2y \implies g(y) = 1 + 2y^2 + 2y.
\]

The local extrema for \(g\) are the points \(y\) solutions of \(g'(y) = 0\), that is, \(4y + 2 = 0\), so we conclude \(y = -1/2\) and \(x = \pm \sqrt{1 - 1/4} = \pm \sqrt{3}/2\), that is,

\[
\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \implies f\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{3}{4} - \frac{1}{2} = \frac{1}{2}.
\]

Therefore, the absolute extrema are

\[
(0, 1) \text{ absolute maximum}, \quad \left(0, -\frac{1}{3}\right) \text{ absolute minimum}.
\]