1. (20 points) Find the center and the radius of the sphere \( x^2 + y^2 + z^2 + 3x - 4y = 0 \). Sketch a qualitative picture of the sphere in a Cartesian coordinate system in \( \mathbb{R}^3 \).

**SOLUTION:** We complete squares to find the center point and the radius of the sphere.

\[
0 = x^2 + y^2 + z^2 + 3x - 4y \\
= \left[ x^2 + 2\left(\frac{3}{2}\right)x + \frac{9}{4}\right] - \frac{9}{4} + \left[ y^2 - 2(2y) + 4\right] - 4 + z^2 \\
= \left( x + \frac{3}{2} \right)^2 + (y - 2)^2 + z^2 - \left( \frac{9 + 16}{4} \right) \\
\Rightarrow \left( x + \frac{3}{2} \right)^2 + (y - 2)^2 + z^2 = \left( \frac{5}{2} \right)^2.
\]

We conclude that the sphere center point and radius are, respectively,

\[
P_0 = \left( -\frac{3}{2}, 2, 0 \right), \quad r = \frac{5}{2}.
\]
2. (10 points) Find the components in a 2-dimensional Cartesian coordinate system of a force vector with magnitude $|\mathbf{F}| = 3$ and having an angle $\theta = \pi/3$ with the positive horizontal axis.

Solution: If the vector $\mathbf{F}$ has an angle $\theta$ with the positive horizontal axis, then its components in a Cartesian coordinate system $\mathbf{F} = \langle F_x, F_y \rangle$ can be written as follows,

$$F_x = |\mathbf{F}| \cos(\theta), \quad F_y = |\mathbf{F}| \sin(\theta).$$

Therefore

$$F_x = 3 \cos(\pi/3) = \frac{3}{2}, \quad F_y = 3 \sin(\pi/3) = \frac{3\sqrt{3}}{2} \Rightarrow \mathbf{F} = \frac{3}{2} \langle 1, \sqrt{3} \rangle.$$ 

\[ \triangle \]

3. (a) (5 points) Find a unit vector in the direction of $\mathbf{v} = \langle -1, 2, 1 \rangle$.

(b) (5 points) Find the scalar projection of $\mathbf{w} = \langle 1, 2, 1 \rangle$ onto $\mathbf{v}$.

(c) (5 points) Find the vector projection of $\mathbf{w}$ onto $\mathbf{v}$.

Solution:

(a)

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+4+1}} \langle -1, 2, 1 \rangle \Rightarrow \mathbf{u} = \frac{1}{\sqrt{6}} \langle -1, 2, 1 \rangle.$$ 

(b)

$$P_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{-1 + 4 + 1}{\sqrt{6}} \Rightarrow P_{\mathbf{v}}(\mathbf{w}) = \frac{4}{\sqrt{6}}.$$ 

(c)

$$P_{\mathbf{v}}(\mathbf{w}) = P_{\mathbf{v}}(\mathbf{w}) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{4}{\sqrt{6}} \frac{1}{\sqrt{6}} \langle -1, 2, 1 \rangle \Rightarrow P_{\mathbf{v}}(\mathbf{w}) = \frac{2}{3} \langle -1, 2, 1 \rangle.$$
4. (a) (10 points) Find the intersection of the lines

\[
\begin{align*}
x &= t, & x &= 2s + 2, \\
y &= -t + 2, & y &= s + 3, \\
z &= t + 1, & z &= 5s + 6.
\end{align*}
\]

(b) (10 points) Find the equation of the plane determined by these lines.

**Solution:**

(a) The intersection point of these lines is solution of the equations:

\[
\begin{align*}
t &= 2s + 2, & -t + 2 &= s + 3, & t + 1 &= 5s + 6,
\end{align*}
\]

Substitute \( t \) from the first equation into the second one:

\[
- (2s + 2) + 2 = s + 3 \quad \Rightarrow \quad -2s = s + 3 \quad \Rightarrow \quad s = -1 \quad \Rightarrow \quad t = 0.
\]

These lines intersect because these values of \( t = 0 \) and \( s = -1 \) satisfy the third equation above: \( 0 + 1 = 5(-1) + 6 \). Therefore, the intersection point is \( P_0 = (0, 2, 1) \).

(b) The lines intersect, so they determine a plane. The equation of the plane is fixed by a point in the plane and the normal vector. A point in the plane is the intersection point \( P_0 = (0, 2, 1) \). (Any point on either line is ok, but we have already computed the intersection, so we use that point.) The normal vector \( n \) to the plane is the cross product of the line tangent vectors. We first rewrite the parametric equations for the lines into the vector equations:

\[
\begin{align*}
\hat{r}(t) &= \langle 0, 2, 1 \rangle + \langle 1, -1, 1 \rangle t, & \hat{r}(s) &= \langle 2, 3, 6 \rangle + \langle 2, 1, 5 \rangle s.
\end{align*}
\]

Therefore, a normal vector \( n \) to the plane is

\[
\begin{align*}
n &= v \times w, & v &= \langle 1, -1, 1 \rangle, & w &= \langle 2, 1, 5 \rangle,
\end{align*}
\]

that is,

\[
\begin{vmatrix}
i & j & k \\
1 & -1 & 1 \\
2 & 1 & 5
\end{vmatrix} = \langle (-5 - 1), -(5 - 2), (1 + 2) \rangle = \langle -6, -3, 3 \rangle \quad \Rightarrow \quad n = \langle -6, -3, 3 \rangle.
\]

A simpler solution is \( \hat{n} = \langle -2, -1, 1 \rangle \). We conclude that the equation of the plane is

\[
-2(x - 0) - (y - 2) + (z - 1) = 0 \quad \Rightarrow \quad -2x - y + z = -1.
\]

\[\langle \rangle\]

5. (a) (10 points) Find the cosine of the angle between the planes $2x - 3y + z = 1$ and $-x - 3y + 2z = 5$. 

(b) (10 points) Find the vector equation of the line of intersection of the two planes given in (a).

Solution:

(a) The angle between the plane is the angle between their normal vectors. These 
vectors are: $\mathbf{n} = \langle 2, -3, 1 \rangle$ and $\mathbf{N} = \langle -1, -3, 2 \rangle$. The cosine of the angle is:

$$\cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}| |\mathbf{N}|} = \frac{-2 + 9 + 2}{\sqrt{4 + 9 + 1} \sqrt{1 + 9 + 4}} \Rightarrow \cos(\theta) = \frac{9}{14}.$$

(b) The vector tangent to the line is $\mathbf{v} = \mathbf{n} \times \mathbf{N}$, that is,

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -1 & -3 & 2 \end{vmatrix} = \langle (-6 + 3), -(4 + 1), (-6 - 3) \rangle = \langle -3, -5, -9 \rangle$$

A simpler tangent vector is $\hat{\mathbf{v}} = \langle 3, 5, 9 \rangle$. We now need to find a point in the intersection of the two planes. We first substitute the $z$ coordinate from one plane into the other one:

$$z = 1 - 2x + 3y \quad \Rightarrow \quad -x - 3y + 2(1 - 2x + 3y) = 5 \quad \Rightarrow \quad -5x + 3y = 3.$$ 

We find a simple solution $x = 0$, then $y = 1$, and then $z = 4$. The intersection point is $P_1 = (0, 1, 4)$. The equation of the line is 

$$\mathbf{r}(t) = \langle 0, 1, 4 \rangle + \langle 3, 5, 9 \rangle t.$$ 

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6. (15 points) Sketch a graph of the surface \( x^2 - y^2 + \frac{z^2}{4} = 0 \).

**Solution:** We can write the equation above as

\[
y^2 = x^2 + \frac{z^2}{4} \quad \Rightarrow \quad y = \pm \sqrt{x^2 + \frac{z^2}{4}}.
\]

Therefore, the surface is an elliptical cone along the \( y \) axis.

\[
\begin{array}{c}
\text{x} \\
\text{z} \\
\text{y}
\end{array}
\]