1. (20 points) Find the solution \( y(t) \) to the initial value problem

\[ t y' + (1 + t) y = 3, \quad y(1) = 0. \]

**Solution:** Write the equation in the following way:

\[ y' + \left( \frac{1}{t} + 1 \right) y = \frac{3}{t}. \]

This is an equation of the form \( y' + a(t)y = b(t) \) and the solution can be computed as follows: First find the integrating factor,

\[ a(t) = 1 + \frac{1}{t} \quad \Rightarrow \quad A(t) = \int a(t)dt = t + \ln(t) \quad \Rightarrow \quad \mu(t) = e^{A(t)} = te^t. \]

Second find the solution,

\[ y(t) = \frac{1}{te^t} \left[ c_0 + \int (te^t) \frac{3}{t} dt \right] = \frac{e^{-t}}{t} (c_0 + 3e^t) = \frac{c_0}{t} e^{-t} + \frac{3}{t} e^t. \]

The initial condition \( y(1) = 0 \) implies \( 0 = c_0 e^{-1} + 3 \), that is \( c_0 = -3e \). Therefore,

\[ y(t) = -3 \frac{e}{t} e^{-t} + \frac{3}{t} e^t \quad \Rightarrow \quad y(t) = \frac{3}{t} \left( 1 - e^{1-t} \right). \]
2. (a) (15 points) Compute an **implicit** expression for the solution \( y(t) \) to the initial value problem

\[
y' = \frac{e^{2t} - e^{-2t}}{4 + 3y}, \quad y(0) = 0.
\]

(b) (5 points) Find the **explicit** expression for the solution found in part (2a).

**Solution:**

(a) The differential equation is separable, so we find the solution as follows:

\[
\int (4 + 3y) dy = \int \left( e^{2t} - e^{-2t} \right) dt + c_0 \quad \Rightarrow \quad 4y + \frac{3}{2} y^2 = \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t} + c_0.
\]

From the initial condition one obtains

\[
0 = \frac{1}{2} + \frac{1}{2} + c_0 \quad \Rightarrow \quad c_0 = -1.
\]

Therefore, the implicit expression for the solution \( y(t) \) of the equation above is

\[
\frac{3}{2} y^2(t) + 4y(t) - \frac{1}{2} e^{2t} - \frac{1}{2} e^{-2t} + 1 = 0.
\]

(b) The explicit expression of the solution above can be obtained finding the appropriate root of the equation above. Both roots are given by:

\[
y_\pm(t) = \frac{1}{3} \left[ -4 \pm \sqrt{16 + 3(e^{2t} + e^{-2t})} - 6 \right].
\]

However, only the function \( y_+(t) \) satisfies the initial condition, then the explicit expression for the solution is

\[
y(t) = \frac{1}{3} \left[ -4 + \sqrt{16 + 3(e^{2t} + e^{-2t})} - 6 \right].
\]
3. (30 points) Find all solutions $y(x)$ of the differential equation

$$\left(\frac{5y^3}{x^2} + \frac{3}{x}\right)y' + \frac{3y}{x^2} + 5x = 0.$$ 

You can leave the solution $y(x)$ expressed in implicit form.

**Solution:** We first verify whether the equation above is exact or not:

$$N = \frac{5y^3}{x^2} + \frac{3}{x} \Rightarrow \quad N_x = -\frac{10y^3}{x^3} - \frac{3}{x^2}$$

$$M = \frac{3y}{x^2} + 5x \Rightarrow \quad M_y = \frac{3}{x^2}.$$ 

Therefore the equation above is not exact. We now verify whether there exists an integrating factor $\mu(x)$ that convert the equation into an exact equation:

$$\frac{\mu_x(x)}{\mu(x)} = \frac{1}{N} (M_y - N_x) = \frac{1}{\frac{5y^3}{x^2} + \frac{3}{x}} \left[ \frac{3}{x^2} + \frac{10y^3}{x^3} + \frac{3}{x^2} \right] = \frac{2}{x} \left[ \frac{5y^3}{x^2} + \frac{3}{x} \right] = \frac{2}{x}.$$ 

Therefore, the integrating factor is the solution of the equation $\mu_x/\mu = 2/x$ which is given by $\mu(x) = x^2$. The equation to solve can be transformed into the following exact equation:

$$(5y^3 + 3x) y' + (3y + 5x^3) = 0 \quad \Rightarrow \quad \begin{cases} N = 5y^3 + 3x \Rightarrow N_x = 3, \\ M = 3y + 5x^3 \Rightarrow M_y = 3. \end{cases}$$

Then, the solution can be computed solving the following equations:

$$\phi_y = 5y^3 + 3x \quad \Rightarrow \quad \phi = \frac{5}{4}y^4 + 3xy + g(x),$$

$$\phi_x = 3y + g_x = M = 3y + 5x^3 \quad \Rightarrow \quad g_x = 5x^3 \quad \Rightarrow \quad g(x) = \frac{5}{4}x^4 + c_0.$$ 

So we find that $\phi(x, y) = \frac{5}{4}y^4 + 3xy + \frac{5}{4}x^4 + c_0$. where $c_0$ is a constant. Therefore, the solution $y(x)$ is given implicitly by the expression:

$$\frac{5}{4}y^4(x) + 3xy(x) + \frac{5}{4}x^4 + c_0 = 0.$$
4. (a) (10 points) Find the general solution $y(t)$ of the differential equation:

$$y'' + 2y' - 3y = 0.$$ 

(b) (10 points) Find the particular solutions $y_1(t)$ and $y_2(t)$ of the differential equation given in part (4a) corresponding to the initial conditions:

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

(c) (10 points) Are the solutions $y_1(t)$ and $y_2(t)$ found in part (4b) linearly independent or linearly dependent? Justify your answer, and show your work.

**Solution**

(a) The characteristic equation is the following:

$$r^2 + 2r - 3 = 0 \quad \Rightarrow \quad r = \frac{1}{2}(-2 \pm \sqrt{4 + 12}) = \frac{1}{2}(-2 \pm 4) \quad \Rightarrow \quad r_1 = 1, \quad r_2 = -3.$$ 

Then, the general solution is given by:

$$y(t) = c_1 e^t + c_2 e^{-3t}$$

(b) Given the general solution found in the previous part we can compute two particular solutions $y_1$ and $y_2$ as follows:

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \Rightarrow \quad 1 = c_1 + c_2, \quad 0 = c_1 - 3c_2 \quad \Rightarrow \quad c_1 = \frac{3}{4}, \quad c_2 = \frac{1}{4};$$

therefore $y_1(t) = (3e^t + e^{-3t})/4$. Analogously, we compute the second solution $y_2(t)$.

$$y_2(0) = 0, \quad y_2'(0) = 1 \quad \Rightarrow \quad 0 = c_1 + c_2, \quad 1 = c_1 - 3c_2 \quad \Rightarrow \quad c_1 = \frac{1}{4}, \quad c_2 = -\frac{1}{4};$$

therefore $y_2(t) = (e^t - e^{-3t})/4$.

(c) The solutions $y_1(t)$ and $y_2(t)$ are linearly independent since their Wronskian is different from zero in at least one point. If we choose that point to be $t = 0$, then it is clear that:

$$W_{y_1,y_2}(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

therefore, $y_1(t), y_2(t)$ are l.i.