1. (30 points) Use the substitution \( v = y^{-2} \) to find the solution to the initial value problem

\[
t^2 y' + 2ty - 5y^3 = 0, \quad y(1) = \frac{1}{\sqrt{2}}, \quad t \geq 1.
\]

**Solution:** Divide the equation by \( y^3 \),

\[
t^2 \frac{y'}{y^3} + 2t \frac{1}{y^2} = 5,
\]

then introduce the substitution \( v = y^{-2} \), which implies \( v' = -2y'/y^3 \), into the equation above:

\[
-\frac{t^2}{2} v' + 2t v = 5 \quad \Rightarrow \quad v' - \frac{4}{t} v = -\frac{10}{t^2}.
\]

The last equation is of the form \( v' + a(t)v = b(t) \) and can be integrated as follows:

\[
a(t) = -\frac{4}{t} \quad \Rightarrow \quad A(t) = -4 \ln(t) = \ln(t^{-4}) \quad \Rightarrow \quad \mu(t) = e^{A(t)} = t^{-4} = \frac{1}{t^4},
\]

\[
v(t) = \frac{1}{\mu(t)} \left[ c_0 + \int \mu(s)b(s) \, ds \right] = t^4 \left[ c_0 - \int \frac{10}{s^4} \frac{1}{s^2} \, ds \right]
\]

\[
v(t) = c_0 t^4 - 10 t^4 \int s^{-6} \, ds = c_0 t^4 - 10 t^4 \frac{t^{-5}}{-5} = c_0 t^4 + \frac{2}{t}.
\]

The initial condition \( y(1) = 1/\sqrt{2} \) implies \( v(1) = 2 \), so the constant \( c_0 \) is given by \( 2 = v(1) = c_0 + 2 \), so \( c_0 = 0 \). Then, \( v(t) = 2/t \), so this implies that the solution \( y(t) \) is

\[
y(t) = \sqrt{\frac{t}{2}}.
\]
2. (30 points) Sometimes a constant equilibrium solution has the property that solutions on one side of the equilibrium approach it, while solutions on the other side of the equilibrium depart from it. Such equilibria are called *semistable*. For the following equation, determine the constant equilibrium solutions, classify each as asymptotically stable, unstable or semistable, and sketch several graphs of solutions in the $ty$-plane corresponding to different initial conditions $y_0$. Sketch the correct concavity of the solution graphs in the $ty$-plane.

$$y' = y(y - 7)^2, \quad -\infty < y_0 < \infty$$

**Solution:** The function $f(y) = y(y - 7)^2$ has two roots at the points $y_1 = 0$ and $y_2 = 7$. Its derivative is given by $f'(y) = (y - 7)(3y - 7)$, therefore it vanishes at $y_2 = 7$ and $y_3 = 7/3$. The graphs of both functions is given in Fig. 1. The sign of $f$ determines the intervals where $y(t)$ solution of the equation $y' = f(y)$ is increasing ($f > 0$) or decreasing ($f < 0$). The concavity of $y(t)$ is given by the sign of $y'' = f(y)f'(y)$. The resulting graph of several solutions $y(t)$ is also given in Fig. 1.

![Figure 1: The graphs of functions $f$, $f'$ and the solutions $y$ for different initial conditions.](image-url)
3. (40 points) Determine whether the following differential equations are exact. Only in the case that the equation is exact, find the (implicit) solution.

(a) \((xe^y + x \sin(xy))'y + xe^y + y \sin(xy) = 0\).

(b) \((6xy + x^3) y' + 3y^2 + 3x^2y = 0\).

**Solution:**

(a)

\[\begin{align*}
N &= xe^y + x \sin(xy) \quad \Rightarrow \quad N_x = e^y + \sin(xy) + xy \cos(xy) \\
M &= xe^y + y \sin(xy) \quad \Rightarrow \quad M_y = xe^y + \sin(xy) + yx \cos(xy),
\end{align*}\]

therefore, the equation is *not* exact.

(b)

\[\begin{align*}
N &= 6xy + x^3 \quad \Rightarrow \quad N_x = 6y + 3x^2 \\
M &= 3y^2 + 3x^2y \quad \Rightarrow \quad M_y = 6y + 3x^2,
\end{align*}\]

therefore, the equation is exact. Then, it can be integrated as follows:

\[\phi_y = N = 6xy + x^3 \quad \Rightarrow \quad \phi = 3xy^2 + x^3y + g(x).\]

Therefore we can compute \(\phi_x = 3y^2 + 3x^2y + g'(x)\). However, we also know that \(\phi_x = M = 3y^2 + 3x^2y\), so we conclude that \(g'(x) = 0\), and so \(g(x) = c_0\), a constant. Then we have \(\phi(x, y) = 3xy^2 + x^3y + c_0\), so the implicit expression for the solutions \(y(x)\) is:

\[3xy^2(x) + x^3y(x) + c_0 = 0\]