Integrals of functions on infinite domains

- Review: Improper integrals type I.
- Type II: Three main possibilities.
- Limit of an infinite sequence.

Generalizations of $\int_a^b f(x) \, dx$ in $I = [a, b]$

Integrals on infinite domains are called improper integrals of type I

- Type I: The interval is infinite: $I = (-\infty, b]$, or $I = [a, \infty)$ or $I = (-\infty, \infty)$.

Integrals of divergent functions on finite domains are called improper integrals of type II.

- Type II: $f(x)$ is not bounded at one or more points in $[a, b]$. ($f(x)$ can have a vertical asymptote in $[a, b]$.)
Type II: Vertical asymptote at $b$

Possibility (a):

**Definition 1** If $f(x)$ is continuous in $[a,b)$ then

$$
\int_a^b f(t) \, dt = \lim_{x \to b^-} \int_a^x f(t) \, dt.
$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.

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Type II: Vertical asymptote at $a$

Possibility (b):

**Definition 2** If $f(x)$ is continuous in $(a,b]$ then

$$
\int_{a+}^b f(t) \, dt = \lim_{x \to a^+} \int_x^b f(t) \, dt.
$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.
Type II: Vertical asymptote in the interior

Possibility (c):

**Definition 3** If \( f(x) \) has a vertical asymptote at \( c \in (a, b) \), then

\[
\int_a^b f(t) \, dt = \int_a^c f(t) \, dt + \int_c^b f(t) \, dt
\]

provided that both integrals in the right hand side are convergent.

Comparison theorem: Type I (a) case

**Theorem 1** Let \( f(x), g(x) \) be continuous functions for \( x \geq a \) and such that \( 0 \leq g(x) \leq f(x) \). Then:

- If \( \int_a^\infty f(x) \, dx \) converges \( \Rightarrow \int_a^\infty g(x) \, dx \) converges.
- If \( \int_a^\infty g(x) \, dx \) diverges \( \Rightarrow \int_a^\infty f(x) \, dx \) diverges.

There are analogous versions for all other cases.
A sequence is a function whose domain are the positive integers

**Definition 4** If for every positive integer $n$ there is associated a real or complex number $a_n$, then the ordered set

$$a_1, a_2, \ldots, a_n, \ldots$$

is called an infinite sequence. It is denoted as $\{a_n\}$.

**Definition 5** A function $f : \mathbb{Z}^+ \to \mathbb{R}$ (or $\mathbb{C}$) is called an infinite sequence.

The limits of sequences is the same as in functions of real numbers

**Definition 6** The sequence $\{a_n\}$ is said to have the limit $L$ is for all $\epsilon > 0$ there exists a number $N > 0$ such that

$$|a_n - L| < \epsilon, \quad \text{for all } n \geq N.$$ 

In this case we say $\lim_{n \to \infty} a_n = n$ or $a_n \to L$ as $n \to \infty$. We say that the sequence converges.

Otherwise, we say that the sequence diverges.
Increasing-decreasing and bounded above-below are important classes of sequences

- A sequence \( \{a_n\} \) is said to be increasing \( \iff \) \( a_n < a_{n+1} \) for all \( n \geq 1 \).
- A sequence \( \{a_n\} \) is said to be decreasing \( \iff \) \( a_{n+1} < a_n \) for all \( n \geq 1 \).

- A sequence \( \{a_n\} \) is said to be bounded above \( \iff \) exists \( M > 0 \) such that \( a_n < M \) for all \( n \geq 1 \).
- A sequence \( \{a_n\} \) is said to be bounded below \( \iff \) exists \( m > 0 \) such that \( m < a_n \) for all \( n \geq 1 \).

Important tool to show that a sequence converges

- If \( \{a_n\} \) is increasing and bounded above then converges.
- If \( \{a_n\} \) is decreasing and bounded below then converges.