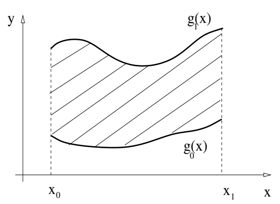
Slide 1

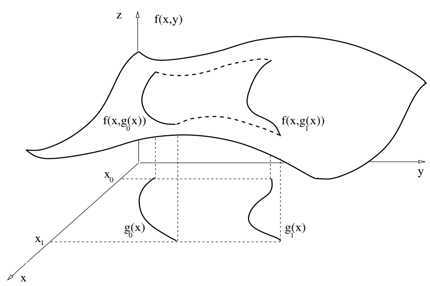
Double integrals on regions

- Regions in Cartesian coordinates (Sec. 15.3)
  - Type I: Regions functions $y(x)$.
  - Type II: Regions functions $x(y)$.

Slide 2

Regions in Cartesian coordinates $y(x)$: Type I





Slide 3

Regions in Cartesian coordinates $y(x)$: Type I

Theorem 1 *Let $g_0(x)$, $g_1(x)$ be two continuous functions defined on an interval $[x_0, x_1]$, and such that $g_0(x) \leq g_1(x)$. Let $f(x, y)$ be a continuous function in*

$$D = \{(x, y) \in I\!R^2 : x_0 \leq x \leq x_1, \quad g_0(x) \leq y \leq g_1(x)\}.$$

Then, the integral of $f(x, y)$ in D is given by

$$\int\int_D f(x, y)\, dxdy = \int_{x_0}^{x_1} \left[\int_{g_0(x)}^{g_1(x)} f(x, y) dy \right] dx.$$

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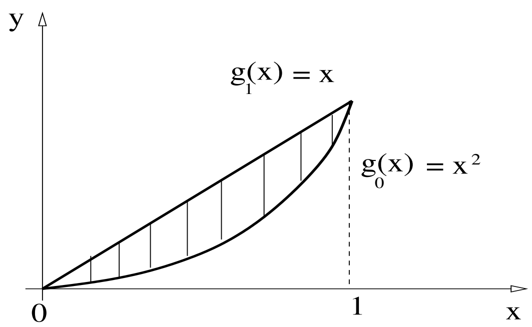
Cartesian Type I: Find the $\int\int_D f(x, y)\, dxdy$ for $f(x, y) = x^2 + y^2$, on $D = \{(x, y) \in I\!R^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}$
.



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$$
\begin{aligned}
\int\int_D f(x,y)\,dxdy &= \int_0^1 \left[\int_{x^2}^{x} (x^2+y^2)dy\right] dx,\\
&= \int_0^1 \left[x^2\left(y|_{x^2}^{x}\right) + \frac{1}{3}\left(y^3|_{x^2}^{x}\right)\right] dx,\\
&= \int_0^1 \left[x^2(x-x^2) + \frac{1}{3}(x^3-x^6)\right] dx,\\
&= \int_0^1 \left[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6\right] dx,\\
&= \left[\frac{1}{4}x^4 - \frac{1}{5}x^5 + \frac{1}{12}x^4 - \frac{1}{21}x^7\right]\Big|_0^1,\\
&= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{3\times 5\times 7}.
\end{aligned}
$$

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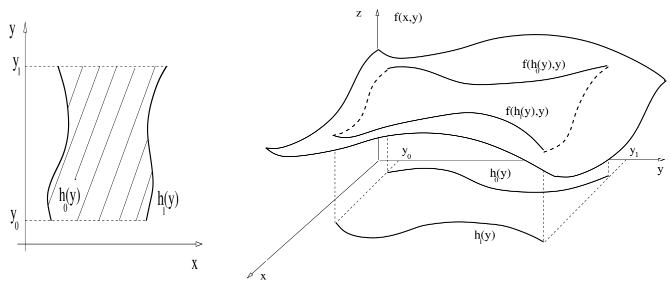
Regions in Cartesian coordinates $x(y)$: Type II



Slide 7

Regions in Cartesian coordinates $x(y)$: Type II

Theorem 2 *Let $h_0(y)$, $h_1(y)$ be two continuous functions defined on an interval $[y_0, y_1]$, and such that $h_0(y) \le h_1(y)$. Let $f(x, y)$ be a continuous function in*

$$D = \{(x, y) \in I\!R^2 : h_0(y) \le x \le h_1(y), \quad y_0 \le y \le y_1\}.$$

Then, the integral of $f(x, y)$ in D is given by

$$\int\int_D f(x, y)\, dxdy = \int_{y_0}^{y_1} \left[\int_{h_0(y)}^{h_1(y)} f(x, y) dx\right] dy.$$

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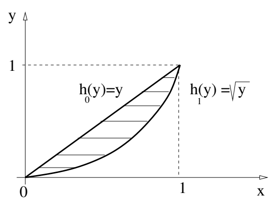
Cartesian Type II: Find the $\int\int_D f(x, y)\, dxdy$ for $f(x, y) = x^2 + y^2$, on $D = \{(x, y) \in I\!R^2 : 0 \le x \le 1, \quad x^2 \le y \le x\}$
.



Notice that $h_0(y) = y$, and $h_1(y) = \sqrt{y}$. Then,

$$D = \{(x, y) \in I\!R^2 : h_0(y) = y \le x \le h_1(y) = \sqrt{y}, \quad y_0 \le y \le y_1\}.$$

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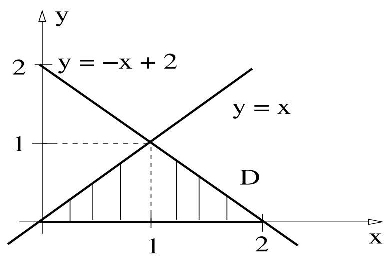
$$\int\int_D f(x,y)\,dxdy = \int_0^1 \left[\int_y^{\sqrt{y}} (x^2+y^2)dx\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}\left(x^3|_y^{\sqrt{y}}\right) + y^2\left(x|_y^{\sqrt{y}}\right)\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}(y^{3/2}-y^3) + y^2(y^{1/2}-y)\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\right] dy,$$
$$= \left[\frac{1}{3}\frac{2}{5}y^{5/2} - \frac{1}{3}\frac{1}{4}y^4 + \frac{2}{7}y^{7/2} - \frac{1}{4}y^4\right]\Big|_0^1,$$
$$= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{3\times 5\times 7}.$$

Slide 10

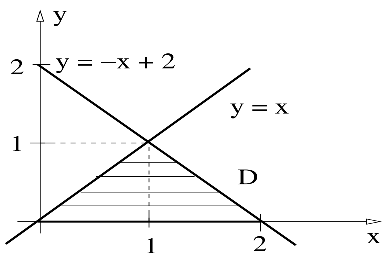
Integrate $f(x,y) = 2xy$ in the region bounded by $y=0$, $y=x$ and $y+x=2$



$$\iint_D f\,dxdy = \int_0^1\int_0^x 2xy\,dy\,dx + \int_1^2\int_0^{2-x} 2xy\,dy\,dx.$$

Slide 11

Integrate $f(x, y) = 2xy$ in the region bounded by $y = 0$, $y = x$ and $y + x = 2$



$$\iint_D f\,dxdy = \int_0^1 \int_y^{2-y} 2xy\,dx\,dy.$$

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Find the $\int\int_D f(x, y)\,dxdy$ for $f(x, y) = 1$, and $D = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$

As type I, then,

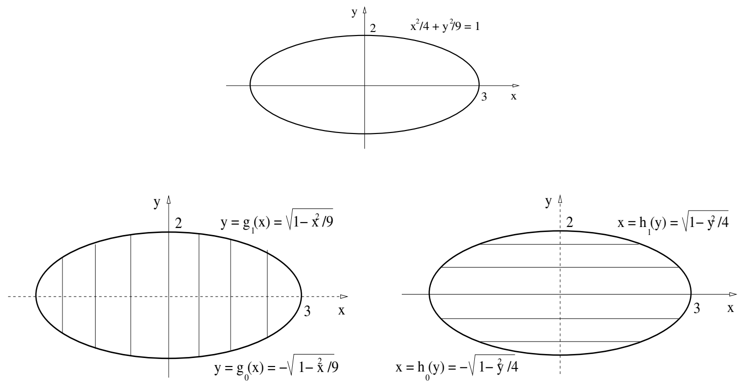
$$g_1(x) = 3\sqrt{1 - y^2/4}, \quad g_0(x) = -3\sqrt{1 - y^2/4}.$$

As type II, then,

$$h_1(x) = 2\sqrt{1 - x^2/9}, \quad h_0(y) = -2\sqrt{1 - x^2/9}.$$

Slide 13

Find the $\int\int_D f(x,y)\,dxdy$ **for** $f(x,y)=1$**, and**
$D = \left\{ (x,y) \in I\!R^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}$



Slide 14

Double integrals on regions in polar coordinates

- Regions in Cartesian coordinates (Sec. 15.4)
  - Type I: Regions functions $r(\theta)$.
  - Type II: Regions functions $\theta(r)$.

Slide 15

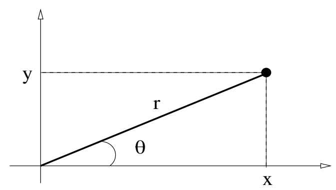
Review of polar coordinates

Definition 1 *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are defined in $\mathbb{R}^2 - \{(0,0)\}$, and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$



Slide 16

Double integrals in polar coordinates on disk sections

Theorem 3 *If $f(r, \theta)$ in continuous in*

$$D = \{(r, \theta) : 0 < r_0 \le r \le r_1, \quad \theta_0 \le \theta \le \theta_1 < 2\pi\},$$

then $\displaystyle\int\int_D f(r,\theta)\, dA = \int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta)\, r\, dr d\theta.$

Disk sections in polar coordinates ↔ rectangular sections in Cartesian coordinates

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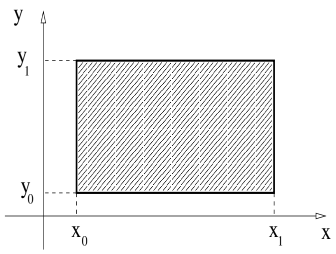
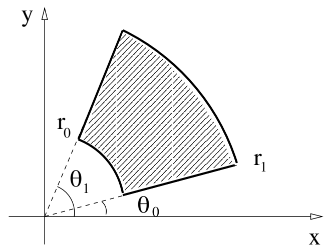
Disk sections in polar coordinates ↔ rectangular sections in Cartesian coordinates





$$x_0 \le x \le x_1, \qquad 0 \le r_0 \le r \le r_1,$$
$$y_0 \le y \le y_1, \qquad 0 \le \theta_0 \le \theta \le \theta_1 \le 2\pi.$$

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Compute the integral of $f(x,y) = x^2 + 2y^2$ on

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y, \quad 0 \le x, \quad 1 \le x^2 + y^2 \le 2\}$

Translate to polar coordinates. $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then

$$f(r,\theta) = r^2 + r^2 \sin^2(\theta).$$

The region D is $D = \left\{(r,\theta) \in \mathbb{R}^2 : 0 \le \theta \le \frac{\pi}{2}, \quad 1 \le r \le \sqrt{2}\right\}$.

$$\int\int_D f(r,\theta) dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 (1 + \sin^2(\theta)) r \, dr d\theta,$$
$$= \left[\int_0^{\pi/2} (1 + \sin^2(\theta))\, d\theta\right] \left[\int_1^{\sqrt{2}} r^3 dr\right],$$

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Example: Continuation

$$\int\int_D f(r,\theta)dA = \left[(\theta|_0^{\pi/2}) + \int_0^{\pi/2} \frac{1}{2}(1-\cos(2\theta))\,d\theta\right]\left[\frac{1}{4}(r^4|_1^{\sqrt{2}})\right],$$

$$= \left[\frac{\pi}{2} + \frac{1}{2}(\theta|_0^{\pi/2}) - \frac{1}{4}(\sin(2\theta)|_0^{\pi/2})\right]\frac{3}{4},$$

$$= \frac{3}{4}\left[\frac{\pi}{2}+\frac{\pi}{4}\right],$$

$$= \frac{9}{16}\pi, \quad \Rightarrow \quad \boxed{\int\int_D f(r,\theta)dA = \frac{9}{16}\pi}.$$

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Integrate $f(x,y) = e^{-(x^2+y^2)}$ on

$D = \{(r,\theta) \in R^2 : \quad 0 \le \theta \le \pi, \quad 0 \le r \le 2\}$

Notice, $f(r,\theta) = e^{-r^2}$, then, $\int\int_D e^{-(x^2+y^2)}dA = \int_0^\pi \int_0^2 e^{-r^2} r\,dr d\theta$, substitute $u = r^2$, then $du = 2r\,dr$, then

$$\int\int_D e^{-(x^2+y^2)}dA = \frac{1}{2}\int_0^\pi \int_0^4 e^{-u} du d\theta$$

$$= \frac{1}{2}\int_0^\pi (-e^{-u}|_0^4)d\theta,$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{e^4}\right),$$

$$\boxed{\int\int_D e^{-(x^2+y^2)}dA = \frac{\pi}{2}\left(1-\frac{1}{e^4}\right)}.$$

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Summarizing, from Cartesian to polar

Theorem 4 *Let $f(x, y)$ be a continuous function on a domain D, where (x, y) represent Cartesian coordinates. Let (r, θ) be polar coordinates. Then the following formula holds,*

$$\int\int_D f(x,y)\,dxdy = \int\int_D f(r\cos(\theta), r\sin(\theta))r\,drd\theta.$$

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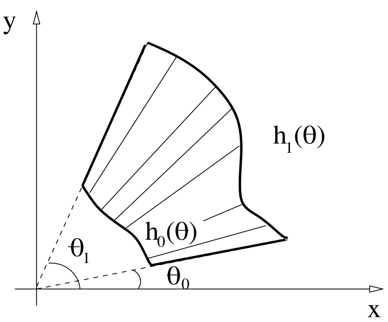
Domains in type I in polar coordinates

$$D = \left\{ (r,\theta) \in I\!R^2 : \begin{array}{c} 0 \leq h_0(\theta) \leq r \leq h_1(\theta), \\ \theta_0 \leq \theta \leq \theta_1 \end{array} \right\}.$$



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Type I in polar coordinates

Theorem 5 *Let $0 < h_0(\theta) \leq h_1(\theta)$ be two continuous functions defined on an interval $[\theta_0, \theta_1]$. Let $f(r,\theta)$ be a continuous function in*

$$D = \left\{ (r,\theta) \in I\!R^2 : \begin{array}{c} 0 \leq h_0(\theta) \leq r \leq h_1(\theta), \\ \theta_0 \leq \theta \leq \theta_1 \end{array} \right\}.$$

Then, the integral of $f(r,\theta)$ in D is given by

$$\int\int_D f(r,\theta)\, dA = \int_{\theta_0}^{\theta_1} \left[\int_{h_0(\theta)}^{h_1(\theta)} f(r,\theta) r\, dr \right] d\theta.$$

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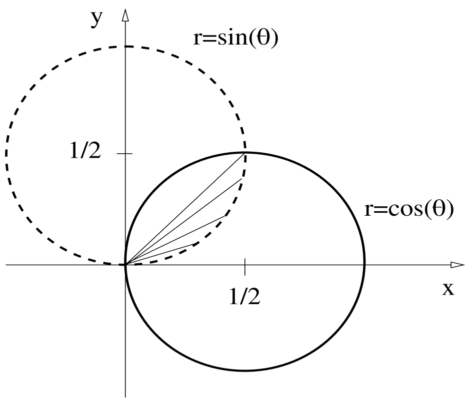
Find the area of intersection of the two circles $r = \cos(\theta)$ **and** $r = \sin(\theta)$



Slide 25

Find the area of intersection of the two circles $r = \cos(\theta)$ **and** $r = \sin(\theta)$

$$\begin{aligned}
A &= \int_0^{\pi/4}\int_0^{\sin(\theta)} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\cos(\theta)} r\,dr\,d\theta,\\
&= \int_0^{\pi/4} \frac{1}{2}\sin^2(\theta)\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2(\theta)\,d\theta,\\
&= \int_0^{\pi/4} \frac{1}{4}[1-\cos(2\theta)]\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{4}[1+\cos(2\theta)]\,d\theta,\\
&= \frac{1}{4}\left[\left(\frac{\pi}{4}-0\right) - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4} + \left(\frac{\pi}{2}-\frac{\pi}{4}\right) + \frac{1}{2}\sin(2\theta)\Big|_{\pi/4}^{\pi/2}\right],\\
&= \frac{1}{4}\left[\frac{\pi}{2} - \left(\frac{1}{2}-0\right) + \left(0-\frac{1}{2}\right)\right],\\
&= \frac{\pi}{8}-\frac{1}{4} \quad \Rightarrow \quad \boxed{A = \frac{1}{8}(\pi-2)}.
\end{aligned}$$

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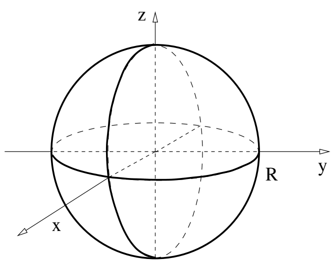
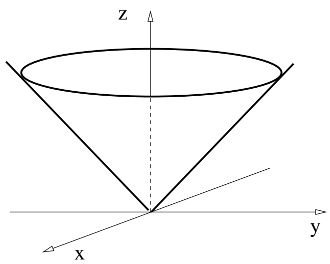
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ **and the cone** $z = \sqrt{x^2+y^2}$





$z = \sqrt{1-r^2}$, $\qquad$ $z = r$.

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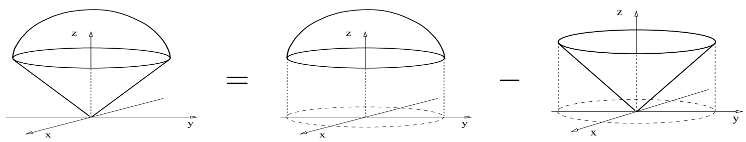
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ **and the cone** $z = \sqrt{x^2 + y^2}$



$$V = \int_0^{2\pi} \int_0^{r_0} \sqrt{1 - r^2}\,(r dr) d\theta - \int_0^{2\pi} \int_0^{r_0} r\,(r dr) d\theta.$$

Slide 28

Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ **and the cone** $z = \sqrt{x^2 + y^2}$

$$V = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{1/\sqrt{2}} \left[\sqrt{1 - r^2} - r\right] r dr\right),$$

$$= 2\pi\left[\int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\, r dr - \int_0^{1/\sqrt{2}} r^2\, dr\right], \qquad u = 1 - r^2$$

$$= 2\pi\left[\frac{1}{2}\int_{1/2}^{1} u^{1/2}\, du - \frac{1}{3} r^3\Big|_0^{1/\sqrt{2}}\right],$$

$$= 2\pi\left[\frac{1}{2}\,\frac{2}{3} u^{3/2}\Big|_{1/2}^{1} - \frac{1}{3}\,\frac{1}{2^{3/2}}\right],$$

$$= \frac{2\pi}{3}\left[1 - \frac{1}{2^{3/2}} - \frac{1}{2^{3/2}}\right], \quad \Rightarrow \quad \boxed{V = \frac{\pi}{3}\left(2 - \sqrt{2}\right)}.$$

Slide 29

Type II in polar coordinates

Theorem 6 *Let* $g_0(r)$, $g_1(r)$ *be two continuous functions defined on an interval* $[r_0, r_1]$, *and such that* $0 < g_0(r) \leq g_1(r) < 2\pi$. *Let* $f(r,\theta)$ *be a continuous function in*

$$D = \{(r,\theta) \in \mathbb{R}^2 : 0 < r_0 \leq r \leq r_1, \\ 0 < \ g_0(r) \leq \theta \leq g_1(r) \ < 2\pi\}.$$

Then, the integral of $f(r,\theta)$ *in* D *is given by*

$$\int\int_D f(r,\theta)\, dA = \int_{r_0}^{r_1} \left[\int_{g_0(r)}^{g_1(r)} f(r,\theta) d\theta \right] r dr.$$