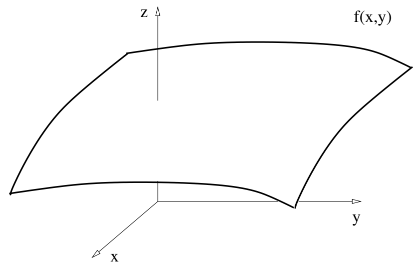
Slide 1

Scalar functions of 2, 3 variables

- Graph and level curves/surfaces. (Sec. 14.1)
- Limits and continuity. (Sec. 14.2)



Slide 2

Scalar functions of 2 variables is denoted as $f(x, y)$

Definition 1 *A scalar function f of two variables (x, y) is a rule that assigns to each ordered pair $(x, y) \in D \subset I\!R^2$ a unique real number, denoted by $f(x, y)$, that is,*

$$f : D \subset I\!R^2 \to R \subset I\!R.$$

Examples:

$$f(x, y) = x^2 + y^2, \quad g(x, y) = \sqrt{x - y}.$$

Slide 3

Compare $f(x, y)$ with $\mathbf{r}(t)$

- Vector valued functions,

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$$
$$t \to \langle x(t), y(t) \rangle$$

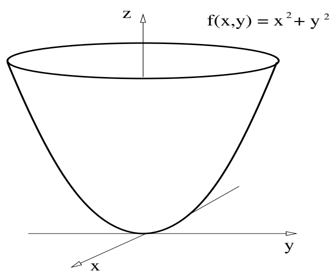
- Scalar function of two variables,

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \to f(x, y).$$

Slide 4

The graph of $f(x, y)$ is a surface in $\mathbb{R}^3$

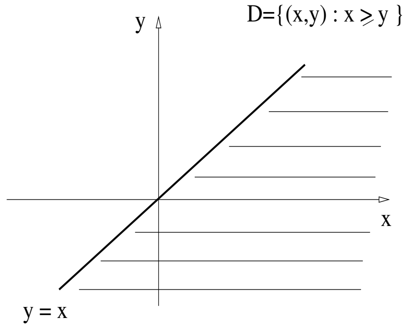
Definition 2 *The graph of a function $f(x, y)$ is the set of all points (x, y, z) in $\mathbb{R}^3$ of the form $(x, y, f(x, y))$.*



Slide 5

The domain of a function may not be the whole plane

Consider $f(x, y) = \sqrt{x - y}$.



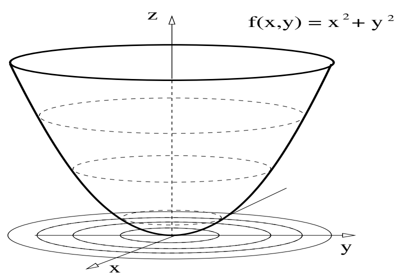
Slide 6

Curves of constant $f(x, y)$ are called level curves

Definition 3 *The level curves of $f(x, y)$ are the curves in in the domain of f, $D \subset \mathbb{R}^2$, solutions of the equation*

$$f(x, y) = k,$$

for $k \in R$, a real constant in the range of f.



Slide 7

Scalar functions of 3 variables are $f(x, y, z)$

Definition 4 *A scalar function f of three variables (x, y, z) is a rule that assigns to each ordered triple $(x, y, z) \in D \subset \mathbb{R}^3$ a unique real number, denoted by $f(x, y, z)$, that is,*

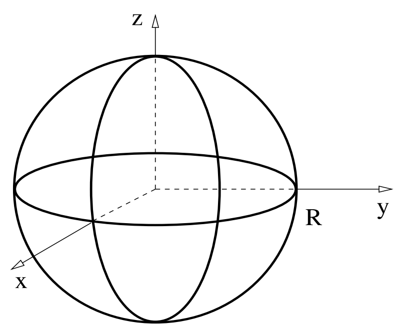
$$f : D \subset \mathbb{R}^3 \to R \subset \mathbb{R}.$$

Example: $f(x, y, z) = x^2 + y^2 + z^2$.

The graph a function $f(x, y, z)$ requires four space dimensions. We cannot picture such graph

Slide 8

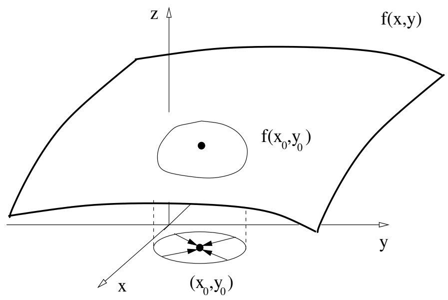
Level curves can be generalized from $f(x, y)$ to $f(x, y, z)$. In this case they are called level surfaces



$$R^2 = f(x, y, z) = x^2 + y^2 + z^2.$$

Slide 9

The function $f(x,y)$ has the number L as limiting value at the point (x_0, y_0) roughly means:



that for all points (x,y) near (x_0, y_0) the value of $f(x,y)$ differs little from L

Slide 10

The definition of limit requires the notion of distance in the plane

Definition 5 *Given a function $f(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$, we write*

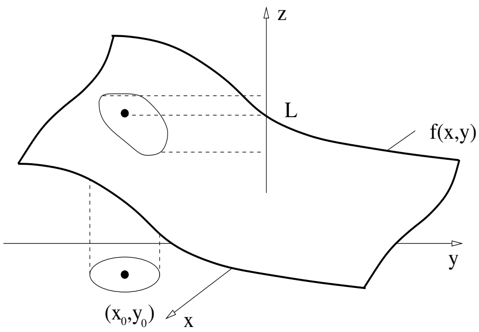
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

if and only if for all $(x,y) \in D$ close enough in distance to (x_0, y_0) the values of $f(x,y)$ approaches L.

Slide 11

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

for all points (x,y) near (x_0,y_0) the value of $f(x,y)$ differs little from L



Slide 12

Limit laws for $f(x)$ also holds for functions $f(x,y)$

If the limits $\lim_{\mathbf{x}\to\mathbf{x}_0} f$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g$ exist, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0}(f \pm g) = \left(\lim_{\mathbf{x}\to\mathbf{x}_0} f\right) \pm \left(\lim_{\mathbf{x}\to\mathbf{x}_0} g\right),$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0}(fg) = \left(\lim_{\mathbf{x}\to\mathbf{x}_0} f\right)\left(\lim_{\mathbf{x}\to\mathbf{x}_0} g\right).$$

Slide 13

Here is a tool to show that a limit does not exist

Theorem 1 *If* $f(x, y) \to L_1$ *along a path* C_1 *as* $(x, y) \to (x_0, y_0)$*, and* $f(x, y) \to L_2$ *along a path* C_2 *as* $(x, y) \to (x_0, y_0)$*, with* $L_1 \neq L_2$*, then*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \quad \textit{does not exist.}$$

When side limits do not agree, the limit does not exist

Slide 14

Here is a tool to show that a limit does exist

Theorem 2 (Squeeze)

Assume $f(x, y) \leq g(x, y) \leq h(x, y)$ *for all* (x, y) *near* (x_0, y_0)*. Also assume*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L = \lim_{(x,y)\to(x_0,y_0)} h(x, y).$$

Then

$$\lim_{(x,y)\to(x_0,y_0)} g(x, y) = L.$$

Example: How to use the side limit theorem.

- Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}. \tag{1}$$

So, the function is $f(x,y) = (3x^2)/(x^2+2y^2)$. Let pick the curve C_1 as the x-axis, that is, $y = 0$. Then,

$$f(x,0) = \frac{3x^2}{x^2} = 3,$$

then

$$\lim_{(x,0)\to(0,0)} f(x,0) = 3.$$

Let us now pick up the curve C_2 as the y-axis, that is, $x = 0$. Then,

$$f(0,y) = 0,$$

then

$$\lim_{(x,0)\to(0,0)} f(x,0) = 0.$$

Therefore, the limit in (1) does not exist.

Notice that in the above example one could compute the limit for arbitrary lines, that is, C_m given by $y = mx$, with m a constant. Then

$$f(x,mx) = \frac{3x^2}{x^2+2m^2x^2} = \frac{3}{1+2m^2},$$

so one has that

$$\lim_{(x,mx)\to(0,0)} f(x,mx) = \frac{3}{1+2m^2}$$

is different for each value of m.

Example: How to use the squeeze theorem.

- Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}. \tag{2}$$

Let us first try the side limit theorem, to try to prove that the limit does not exist. Consider the curves C_m given by $y = mx$, with m a constant. Then

$$f(x,mx) = \frac{x^2mx}{x^2+m^2x^2} = \frac{mx}{1+m^2},$$

so one has that

$$\lim_{(x,mx)\to(0,0)} f(x,mx) = 0, \quad \forall\, m \in \mathbb{R}.$$

Therefore, one cannot conclude that the limit does not exist. However, this argument does not prove that the limit actually exists.This can be done with the squeeze theorem.

First notice that

$$\frac{x^2}{x^2+y^2} \leq 1, \quad \forall\,(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0).$$

(proof: $0 \leq y^2$, then $x^2 \leq (x^2 + y^2)$.) Therefore, one has the inequality

$$-|y| \leq \frac{x^2 y}{x^2 + y^2} \leq |y|, \quad \forall\, (x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0).$$

Then, one knows that $\lim_{y \to 0} |y| = 0$, therefore the squeeze theorem says that

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

Slide 15

Continuous functions have graphs without holes or jumps

Definition 6 *A function* $f(x, y)$ *is continuous at* (x_0, y_0) *if* $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Polynomial functions are continuous in $\mathbb{R}^2$, for example

$$P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.$$

Slide 16

More examples of continuous functions

- Rational functions are continuous on their domain,
$$f(x,y) = \frac{P_n(x,y)}{Q_m(x,y)},$$
for example,
$$f(x,y) = \frac{x^2 + 3y - x^2y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.$$
- Composition of continuous functions are continuous, example
$$f(x,y) = \cos(x^2 + y^2).$$

Slide 17

Partial derivatives

- Definition of Partial derivatives.
- Higher derivatives.
- Examples of differential equations.

Slide 18

To compute partial derivative with respect to x keep y constant

Definition 7 (x-partial derivative) *Let $f : D \subset I\!R^2 \to R \subset I\!R$. The partial derivative of $f(x,y)$ with respect to x at $(a,b) \in D$ is denoted as $f_x(a,b)$ and is given by*

$$f_x(a,b) = \lim_{h \to 0} \frac{1}{h} \left[f(a+h,b) - f(a,b) \right].$$

Slide 19

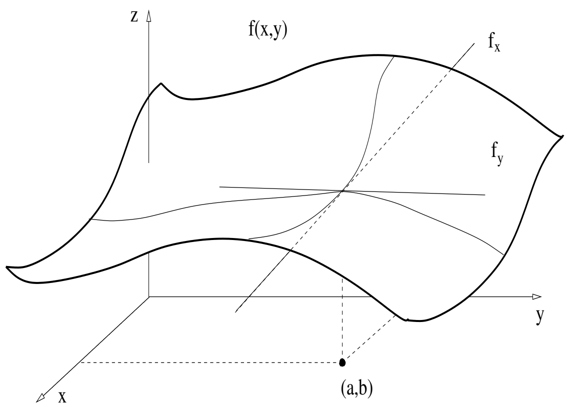
To compute partial derivative with respect to y keep x constant

Definition 8 (y-partial derivative) *Let $f : D \subset I\!R^2 \to R \subset I\!R$. The partial derivative of $f(x,y)$ with respect to y at $(a,b) \in D$ is denoted as $f_y(a,b)$ and is given by*

$$f_y(a,b) = \lim_{h \to 0} \frac{1}{h} \left[f(a,b+h) - f(a,b) \right].$$

Slide 20

Partial derivatives are slopes of lines tangent to the graph of $f(x, y)$



So, to compute the partial derivative of $f(x, y)$ with respect to x at (a, b), one can do the following: First, evaluate the function at $y = b$, that is compute $f(x, b)$; second, compute the usual derivative of single variable functions; evaluate the result at $x = a$, and the result is $f_x(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to x at $(1, 3)$.
  1. $f(x, 3) = x^2 + 9/4$;
  2. $f_x(x, 3) = 2x$;
  3. $f_x(1, 3) = 2$.

To compute the partial derivative of $f(x, y)$ with respect to y at (a, b), one follows the same idea: First, evaluate the function at $x = a$, that is compute $f(a, y)$; second, compute the usual derivative of single variable functions; evaluate the result at $y = b$,, and the result is $f_y(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to y at $(1, 3)$.
  1. $f(1, y) = 1 + y^2/4$;
  2. $f_y(1, y) = y/2$;
  3. $f_y(1, 3) = 3/2$.

Slide 21

Partial derivatives define new functions

Definition 9 *Consider a function* $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$. *The functions partial derivatives of* $f(x,y)$ *are denoted by* $f_x(x,y)$ *and* $f_y(x,y)$*, and are given by the expressions*

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{1}{h}\left[f(x+h,y) - f(x,y)\right], \\ f_y(x,y) &= \lim_{h\to 0} \frac{1}{h}\left[f(x,y+h) - f(x,y)\right]. \end{aligned}$$

Examples:

- $$\begin{aligned} f(x,y) &= ax^2 + by^2 + xy. \\ f_x(x,y) &= 2ax + 0 + y, \\ &= 2ax + y. \\ f_y(x,y) &= 0 + 2by + x, \\ &= 2by + x. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^2 \ln(y), \\ f_x(x,y) &= 2x \ln(y), \\ f_y(x,y) &= \frac{x^2}{y}. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^2 + \frac{y^2}{4}, \\ f_x(x,y) &= 2x, \\ f_y(x,y) &= \frac{y}{2}. \end{aligned}$$

- $$f(x,y) = \frac{2x - y}{x + 2y},$$

$$\begin{aligned} f_x(x,y) &= \frac{2(x+2y)-(2x-y)}{(x+2y)^2}, \\ &= \frac{2x+4y-2x+y}{(x+2y)^2}, \\ &= \frac{5y}{(x+2y)^2}. \\ f_y(x,y) &= \frac{-(x+2y)-(2x-y)2}{(x+2y)^2}, \\ &= \frac{-5x}{(x+2y)^2}. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^3e^{2y}+3y, \\ f_x(x,y) &= 3x^2e^{2y}, \\ f_y(x,y) &= 2x^3e^{2y}+3, \\ f_{yy}(x,y) &= 4x^3e^{2y}, \\ f_{yyy}(x,y) &= 8x^3e^{2y}, \\ f_{xy} &= 6x^2e^{2y}, \\ f_{yx} &= 6x^2e^{2y}. \end{aligned}$$

Slide 22

Higher derivatives of a function $f(x,y)$ are partial derivatives of its partial derivatives

For example, the second partial derivatives of $f(x,y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0}\frac{1}{h}\left[f_x(x+h,y)-f_x(x,y)\right],$$
$$f_{yy}(x,y) = \lim_{h\to 0}\frac{1}{h}\left[f_y(x,y+h)-f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0}\frac{1}{h}\left[f_x(x+h,y)-f_x(x,y)\right],$$
$$f_{yx}(x,y) = \lim_{h\to 0}\frac{1}{h}\left[f_y(x,y+h)-f_y(x,y)\right].$$

Slide 23

Higher partial derivatives sometimes commute

Theorem 3 *Consider a function $f(x, y)$ in a domain D. Assume that f_{xy} and f_{yx} exists and are continuous in D. Then,*

$$f_{xy} = f_{yx}.$$

Slide 24

Differential equations are equations where the unknown is a function

For example, the Laplace equation: Find $\phi(x, y, z) : D \subset \mathbb{R}^3 \to \mathbb{R}$ solution of

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

This equation describes the gravitational effects near a massive object.

and where derivatives of the function enter into the equation

Slide 25

More examples of differential equations

Heat equation: Find a function $T(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R}$ solution of

$$T_t = T_{xx} + T_{yy} + T_{zz}.$$

The heat on a metal is described by this equation. T is the temperature on that object.

Slide 26

More examples of differential equations

Wave equation: Find a function $f(t,x,y,z) : D \subset I\!R^4 \to I\!R$ solution of

$$f_{tt} = f_{xx} + f_{yy} + f_{zz}.$$

The sound in the air is described by this equation. f is the air density.

Exercises:

- Verify that the function $T(t,x) = e^{-t}\sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t,x) = (t-x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$