Print Name: ______________________ **Section Number:** ______

TA Name: ______________________ **Section Time:** ______

Math 20C.
Midterm Exam 2
May 26, 2006

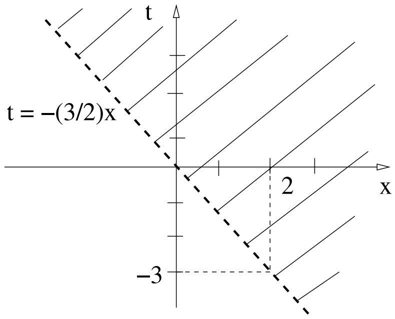
No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all your work.

1. (a) (5 points) Find and sketch the domain of the function $f(x,t) = \ln(3x+2t)$.

   (b) (5 points) Find all possible constants c such that the function $f(x,t)$ above is solution of the wave equation, $f_{tt} - c^2 f_{xx} = 0$.

(a) The argument in the ln function must be positive. then, the domain is

$$\boxed{D = \left\{(x,t) \in I\!R^2 : 3x+2t > 0\right\}}.$$



(b)

$$f_t = \frac{2}{3x+2t}, \qquad f_x = \frac{3}{3x+2t},$$

$$f_{tt} = -\frac{4}{(3x+2t)^2}, \qquad f_{xx} = -\frac{9}{(3x+2t)^2},$$

$$0 = f_{tt} - cf_{xx} = -\frac{4}{(3x+2t)^2} + c^2 \frac{9}{(3x+2t)^2} = \frac{1}{(3x+2t)^2}(-4+9c^2) \quad \Rightarrow$$

$$\Rightarrow \quad 9c^2 = 4, \quad \Rightarrow \boxed{c = \pm\frac{2}{3}}.$$

2. (a) (5 points) Find the direction in which $f(x, y)$ increases the most rapidly, and the directions in which $f(x, y)$ decreases the most rapidly at P_0, and also find the value of the directional derivative of $f(x, y)$ at P_0 along these directions, where

$$f(x, y) = x^3 e^{-2y}, \quad \text{and} \quad P_0 = (1, 0).$$

(b) (5 points) Find the directional derivative of $f(x, y)$ above at the point P_0 in the direction given by $\mathbf{v} = \langle 1, -1 \rangle$.

(a) The direction in which f increases the most rapidly is given by ∇f, and the one in which decreases the most rapidly is $-\nabla f$. So,

$$\nabla f(x, y) = \langle 3x^2 e^{-2y}, -2x^3 e^{-2y} \rangle, \quad \Rightarrow \quad \boxed{\nabla f(1, 0) = \langle 3, -2 \rangle}, \quad \boxed{-\nabla f(1, 0) = \langle -3, 2 \rangle}.$$

The value of the directional derivative along these directions is, respectively, $|\nabla f(1, 0)|$ and $-|\nabla f(1, 0)|$, where

$$\boxed{|\nabla f(1, 0)| = \sqrt{9 + 4} = \sqrt{13}}.$$

(b) A unit vector along $\langle 1, -1 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$, then,

$$D_u f(1, 0) = \nabla f(1, 0) \cdot \mathbf{u} = \langle 3, -2 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle = \frac{5}{\sqrt{2}},$$

$$\boxed{D_u f(1, 0) = \frac{5}{\sqrt{2}}}.$$

3. (a) (5 points) Find the tangent plane approximation of $f(x, y) = x\cos(\pi y/2) - y^2 e^{-x}$ at the point $(0, 1)$.

   (b) (5 points) Use the linear approximation computed above to approximate the value of $f(-0.1, 0.9)$.

(a)

$$\begin{aligned} f(x,y) &= x\cos(\pi y/2) - y^2 e^{-x} & f(0,1) &= -1, \\ f_x(x,y) &= \cos(\pi y/2) + y^2 e^{-x} & f_x(0,1) &= \cos(\pi/2) + 1 = 1, \\ f_y(x,y) &= -x\sin(\pi y/2)\frac{\pi}{2} - 2ye^{-x} & f_y(0,1) &= -2, \end{aligned}$$

Then, the linear approximation $L(x, y)$ is given by

$$L(x,y) = (x-0) - 2(y-1) - 1, \quad \Rightarrow \quad \boxed{L(x,y) = x - 2y + 1}.$$

(b) The linear approximation of $f(-0.1, 0.9)$ is $L(-0.1, 0.9)$, which is given by

$$L(-0.1, 0.9) = -0.1 - 2(-0.1) - 1 = -0.1 - 1 = -1.1, \quad \Rightarrow \quad \boxed{L(-0.1, 0.9) = -1.1}.$$

4. (10 points) Find every local and absolute extrema of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$, and indicate which ones are the absolute extrema. In the case of the interior stationary points, decide whether they are local maximum, minimum of saddle points.

We first compute the interior stationary points, which are (x, y) solutions of

$$\nabla f = \langle 2x, 6y + 2 \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 0, \quad y = -\frac{1}{3}.$$

The point $(0, -1/3)$ belongs to the disk $x^2 + y^2 \leq 1$ so we have to decide whether it is a local maximum, minimum or saddle point:

$$f_{xx} = 2, \quad f_{yy} = 6, \quad f_{xy} = 0,$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 12 > 0, \quad f_{xx} > 0 \quad \Rightarrow \quad \boxed{\left(0, -\frac{1}{3}\right) \text{ is a local minimum.}}$$

This point is also a candidate for absolute minimum, so we record the value of f, which is $f(0, -1/3) = -1/3$.

We now look for extreme point on the boundary $x^2 + y^2 = 1$. We use the Lagrange multipliers method, so we introduce the function $g(x, y) = x^2 + y^2 - 1$, and we solve the system:

$$\nabla f = \lambda \nabla g, \quad g = 0.$$

$$\left.\begin{array}{c} \langle 2x, 6y + 2 \rangle = 2\lambda \langle x, y \rangle \\ x^2 + y^2 = 1 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{rl} x = \lambda x & \Rightarrow \quad x(1 - \lambda) = 0, \\ 3y + 1 = \lambda y, & \Rightarrow \quad y(3 - \lambda) = -1. \\ x^2 + y^2 = 1 & \end{array}$$

The equation $x(1 - \lambda) = 0$ has $x = 0$ as a solution, then the constraint says that $y = \pm 1$, so extrema candidate points are $(0, \pm 1)$. The value of f at those points are

$$f(0, 1) = 5, \qquad f(0, -1) = 1.$$

The other solution of that equation, $x(1 - \lambda) = 0$, is $\lambda = 1$, and this information in $y(3 - \lambda) = -1$ implies $2y = -1$, that is, $y = -1/2$. Then the constraint says $x^2 = 1 - 1/4 = 3/4$, so we have $x = \pm\sqrt{3}/2$. The extrema candidate points are then $(\pm\sqrt{3}/2, -1/2)$. The values of f at those points are

$$f\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{3}{4} - 2\frac{1}{2} = \frac{1}{2}.$$

Therefore, the absolute extrema are

$$\boxed{(0, 1) \text{ absolute maximum}}, \qquad \boxed{\left(0, -\frac{1}{3}\right) \text{ absolute minimum}}.$$