Partial derivatives

- Review: Limits and continuity. (Sec. 14.2)
- Definition of Partial derivatives. (Sec. 14.3)
- Higher derivatives.
- Examples of differential equations.

We recall the definition of limit of $f(x, y)$

Let $f(x, y)$ be a scalar function defined for $P = (x, y)$ near $P_0 = (x_0, y_0)$. Let $d_{P_0P} = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ be the distance between $(x, y)$ and $(x_0, y_0)$. We write

$$\lim_{(x,y) \to (x_0,y_0)} f(x, y) = L,$$

to mean that the values of $f(x, y)$ approaches $L$ as the distance $d_{P_0P}$ approaches zero.
In this case \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) \) exists

Compute side limits along \( C_1 \) and \( C_2 \)
Continuous functions have graphs without holes or jumps

**Definition 1** \( f(x, y) \) is continuous at \((x_0, y_0)\) if

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).
\]

Polynomial functions are continuous in \( \mathbb{R}^2 \), for example

\[
P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.
\]

More examples of continuous functions

- Rational functions are continuous on their domain,
  
  \[
f(x, y) = \frac{P_n(x, y)}{Q_m(x, y)},
  \]
  
  for example,
  
  \[
f(x, y) = \frac{x^2 + 3y - x^2y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.
  \]

- Composition of continuous functions are continuous, example
  
  \[
f(x, y) = \cos(x^2 + y^2).
  \]
To compute partial derivative with respect to $x$
keep $y$ constant

**Definition 2 ($x$-partial derivative)** Let
$f : D \subset \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{R}$. The partial derivative of $f(x, y)$
with respect to $x$ at $(a, b) \in D$ is denoted as $f_x(a, b)$ and
is given by
\[
f_x(a, b) = \lim_{h \to 0} \frac{1}{h} \left[ f(a + h, b) - f(a, b) \right].
\]

To compute partial derivative with respect to $y$
keep $x$ constant

**Definition 3 ($y$-partial derivative)** Let
$f : D \subset \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{R}$. The partial derivative of $f(x, y)$
with respect to $y$ at $(a, b) \in D$ is denoted as $f_y(a, b)$ and
is given by
\[
f_y(a, b) = \lim_{h \to 0} \frac{1}{h} \left[ f(a, b + h) - f(a, b) \right].
\]
Partial derivatives are slopes of lines tangent to the graph of $f(x, y)$

So, to compute the partial derivative of $f(x, y)$ with respect to $x$ at $(a, b)$, one can do the following: First, evaluate the function at $y = b$, that is compute $f(x, b)$; second, compute the usual derivative of single variable functions; evaluate the result at $x = a$, and the result is $f_x(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to $x$ at $(1, 3)$.
  1. $f(1, 3) = 1^2 + 3^2/4 = 1 + 9/4$;
  2. $f_x(1, 3) = 2x$;
  3. $f_x(1, 3) = 2$.

To compute the partial derivative of $f(x, y)$ with respect to $y$ at $(a, b)$, one follows the same idea: First, evaluate the function at $x = a$, that is compute $f(a, y)$; second, compute the usual derivative of single variable functions; evaluate the result at $y = b$, and the result is $f_y(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to $y$ at $(1, 3)$.
  1. $f(1, y) = 1 + y^2/4$;
  2. $f_y(1, y) = y/2$;
  3. $f_y(1, 3) = 3/2$. 


Partial derivatives define new functions

**Definition 4** Consider a function

\[ f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}. \]

The functions partial derivatives of \( f(x, y) \) are denoted by \( f_x(x, y) \) and \( f_y(x, y) \), and are given by the expressions

\[
\begin{align*}
  f_x(x, y) &= \lim_{h \to 0} \frac{1}{h} [f(x + h, y) - f(x, y)], \\
  f_y(x, y) &= \lim_{h \to 0} \frac{1}{h} [f(x, y + h) - f(x, y)].
\end{align*}
\]

The partial derivative functions of a paraboloid are planes

\[ f(x, y) = ax^2 + by^2 + xy. \]

\[
\begin{align*}
  f_x(x, y) &= 2ax + 0 + y, \\
  &= 2ax + y. \\
  f_y(x, y) &= 0 + 2by + x, \\
  &= 2by + x.
\end{align*}
\]
The partial derivative functions of a paraboloid are planes

More examples:

\[ f(x, y) = x^2 \ln(y), \]
\[ f_x(x, y) = 2x \ln(y), \]
\[ f_y(x, y) = \frac{x^2}{y}. \]

\[ f(x, y) = x^2 + \frac{y^2}{4}, \]
\[ f_x(x, y) = 2x, \]
\[ f_y(x, y) = \frac{y}{2}. \]

\[ f(x, y) = \frac{2x - y}{x + 2y}, \]
\[ f_x(x, y) = \frac{2(x + 2y) - (2x - y)}{(x + 2y)^2}, \]
\[ = \frac{2x + 4y - 2x + y}{(x + 2y)^2}, \]
\[ = \frac{5y}{(x + 2y)^2}. \]
\[ f_y(x, y) = \frac{-(x + 2y) - (2x - y)^2}{(x + 2y)^2}, \]
\[ = \frac{-5x}{(x + 2y)^2}. \]

\[ f(x, y) = x^3e^{2y} + 3y, \]
\[ f_x(x, y) = 3x^2e^{2y}, \]
\[ f_y(x, y) = 2xe^{2y} + 3, \]
\[ f_{yy}(x, y) = 4xe^{2y}, \]
\[ f_{yyy}(x, y) = 8x^3e^{2y}, \]
\[ f_{xy} = 6x^2e^{2y}, \]
\[ f_{yx} = 6x^2e^{2y}. \]

Higher derivatives of a function \( f(x, y) \) are partial derivatives of its partial derivatives

For example, the second partial derivatives of \( f(x, y) \) are the following:
\[ f_{xx}(x, y) = \lim_{h \to 0} \frac{1}{h} [f_x(x + h, y) - f_x(x, y)], \]
\[ f_{yy}(x, y) = \lim_{h \to 0} \frac{1}{h} [f_y(x, y + h) - f_y(x, y)], \]
\[ f_{xy}(x, y) = \lim_{h \to 0} \frac{1}{h} [f_x(x + h, y) - f_x(x, y)], \]
\[ f_{yx}(x, y) = \lim_{h \to 0} \frac{1}{h} [f_y(x, y + h) - f_y(x, y)]. \]
Higher partial derivatives sometimes commute

**Theorem 1** Consider a function \( f(x, y) \) in a domain \( D \). Assume that \( f_{xy} \) and \( f_{yx} \) exists and are continuous in \( D \). Then,

\[
f_{xy} = f_{yx}.
\]

Differential equations are equations where the unknown is a function

For example, the Laplace equation: Find \( \phi(x, y, z) : \mathbb{R}^3 \to \mathbb{R} \) solution of

\[
\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.
\]

This equation describes the gravitational effects near a massive object.

and where derivatives of the function enter into the equation
More examples of differential equations

Heat equation: Find a function
\( T(t, x, y, z) : D \subset \mathbb{R}^4 \rightarrow \mathbb{R} \) solution of
\[ T_t = T_{xx} + T_{yy} + T_{zz}. \]

The heat on a metal is described by this equation. \( T \) is the temperature on that object.

More examples of differential equations

Wave equation: Find a function
\( f(t, x, y, z) : D \subset \mathbb{R}^4 \rightarrow \mathbb{R} \) solution of
\[ f_{tt} = f_{xx} + f_{yy} + f_{zz}. \]

The sound in the air is described by this equation. \( f \) is the air density.
Exercises:

- Verify that the function $T(t, x) = e^{-t} \sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t, x) = (t - x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$
A function can have partial derivatives at a point and be discontinuous at that point. This is a very bad property for a definition of derivative.

Here is one of such functions, given explicitly:

\[ f(x, y) = \begin{cases} 
2xy/(x^2 + y^2) & (x, y) \neq (0, 0), \\
0 & (x, y) = (0, 0). 
\end{cases} \]

\[ f_x(0, 0) = f_y(0, 0) = 0, \text{ although } f(x, y) \text{ is not continuous at } (0, 0). \]
Recall the following property of the derivative of $f(x)$

**Theorem 2** If $f'(x)$ exists, then $f(x)$ is continuous.

\[
\lim_{h \to 0} [f(x + h) - f(x)] = \lim_{h \to 0} \frac{[f(x + h) - f(x)]}{h} h, \\
= \lim_{h \to 0} f'(x) h = 0.
\]

The analogous claim “If $f_x(x, y)$ and $f_y(x, y)$ exists, then $f(x, y)$ is continuous” is false

One has to define a notion of derivative having the continuity property discussed above.

New definition: A differentiable function must be approximated by a plane.
In the case $f(x)$ this definition says: The function must be approximated by a line

Only for functions $f(x)$ the derivative $f'(x)$ implies the existence of an approximating line $L(x)$

A function of two variables is differentiable at $(x_0, y_0)$ if two conditions hold:

- There exists the plane from its partial derivatives at $(x_0, y_0)$;
- This plane approximates the graph of $f(x, y)$ near $(x_0, y_0)$. 
Here is a rewording of the definition

**Definition 5** The function $f(x, y)$ is differentiable at $(x_0, y_0)$ if

$$f(x, y) = L_{(x_0, y_0)}(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0),$$

where $\epsilon_i(x, y) \to 0$ when $(x, y) \to (x_0, y_0)$, for $i = 1, 2$, and

$$L_{(x_0, y_0)}(x, y) = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) + f(x_0, y_0)$$

This notion of differentiability has the continuity property

**Theorem 3** If $f(x, y)$ is differentiable, then $f(x, y)$ is continuous.

If $f(x, y)$ is differentiable, then $L_{(x_0, y_0)}(x, y)$ is called the linear approximation of $f(x, y)$ at $(x_0, y_0)$. 
The following result is useful to check the differentiability of a function.

**Theorem 4** Consider a function $f(x, y)$. Assume that its partial derivatives $f_x(x, y)$, $f_y(x, y)$ exist at $(x_0, y_0)$ and near $(x_0, y_0)$, and both are continuous functions at $(x_0, y_0)$. Then, $f(x, y)$ is differentiable at $(x_0, y_0)$.

Consider the following exercise:

1. Show that $f(x, y) = \arctan(x + 2y)$ is differentiable at $(1,0)$.
2. Find its linear approximation at $(1,0)$.

$f_x(x, y) = \frac{1}{1 + (x + 2y)^2}$, $f_y(x, y) = \frac{2}{1 + (x + 2y)^2}$.

These functions are continuous in $\mathbb{R}^2$, so $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$.

$L_{(1,0)}(x, y) = f_x(1,0)(x - 1) + f_y(1,0)(y - 0) + f(1,0)$,

where $f(1,0) = \arctan(1) = \pi/4$, $f_x(1,0) = 1/2$, $f_y(1,0) = 1$. Then,

$L_{(1,0)}(x, y) = \frac{1}{2}(x - 1) + y + \frac{\pi}{4}$. 
Second exercise, on linear approximation

• Find the linear approximation of
  \[ f(x, y) = \sqrt{17 - x^2 - 4y^2} \] at \((2, 1)\).

We need three numbers: \(f(2, 1), f_x(2, 1),\) and \(f_y(2, 1)\). Then, we compute the linear approximation by the formula

\[
L_{(2,1)}(x, y) = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1).
\]

The result is: \(f(2, 1) = 3, f_x(2, 1) = -\frac{2}{3},\) and \(f_y(2, 1) = -\frac{4}{3}\). Then the plane is given by

\[
L_{(2,1)}(x, y) = -\frac{2}{3}(x - 2) - \frac{4}{3}(y - 1) + 3.
\]

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\textit{df} \textbf{is a special name for} \(L_{(x_0)}(x) - f(x_0)\)

Single variable case:

\[ df(x) = L_{x_0}(x) - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)dx. \]

We called \((x - x_0) = dx\).  

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\[ df(x, y) \text{ is a special name for } L_{(x_0, y_0)}(x, y) - f(x_0, y_0) \]

Functions of two variables:
\[
df(x, y) = L_{(x_0, y_0)}(x, y) - f(x_0, y_0),
\]
\[
dx = x - x_0, \quad dy = y - y_0.
\]
Then, the formula is easy to remember:
\[
df(x, y) = f_x(x_0, y_0) \, dx + f_y(x_0, y_0) \, dy.
\]

Different names for the same idea: Compute the linear approximation of a differentiable function.

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An exercise on differentials

- Compute the \( df \) of \( f(x, y) = \ln(1 + x^2 + y^2) \) at \((1, 1)\) for \( dx = 0.1, \, dy = 0.2 \).

\[
df(x_0, y_0) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy,
\]
\[
= \frac{2x_0}{1 + x_0^2 + y_0^2}dx + \frac{2y_0}{1 + x_0^2 + y_0^2}dy.
\]
Then,
\[
df(1, 1) = \frac{2}{3} \cdot \frac{1}{10} + \frac{2}{3} \cdot \frac{2}{10},
\]
\[
= \frac{2}{3} \cdot \frac{3}{10},
\]
\[
= \frac{1}{5}.
\]
Another exercise on differentials

- Use differentials to estimate the amount of tin in a closed tin can with internal diameter of 8 cm and height of 12 cm if the tin is 0.04 cm thick.

Data of the problem: \( h_0 = 12 \text{ cm}, \ r_0 = 4 \text{ cm}, \ dr = 0.04 \text{ cm} \) and \( dh = 0.08 \text{ cm} \). Draw a picture of the cylinder.

The function to consider is the volume of the cylinder,
\[ V(r, h) = \pi r^2 h. \]

Then,
\[
dV(r_0, h_0) = V_r(r_0, h_0)dr + V_h(r_0, h_0)dh, \]
\[
= 2\pi r_0h_0dr + \pi r_0^2dh, \]
\[
= 16.1 \text{ cm}. \]

Chain rule and directional derivatives

- Review: Linear approximations. (Sec. 14.4)
- Chain rule. (Sec. 14.5)
- Cases 1 and 2. Examples.
Recall the chain rule for $f(x)$

Given $f(x)$, and $x(t)$ differentiable functions, introduce $z(t) = f(x(t))$. Then, $z(t)$ is differentiable, and

$$\frac{dz}{dt} = \frac{df}{dx}(x(t)) \frac{dx}{dt}(t).$$

Or, using the new notation,

$$z_t(t) = f_x(x(t)) x_t(t).$$

There are many chain rules for $f(x, y)$

Case 1: Given $f(x, y)$ differentiable, and $x(t), y(t)$ differentiable functions of a single variable, then $z(t) = f(x(t), y(t))$ is differentiable and

$$\frac{dz}{dt} = f_x(x(t), y(t)) \frac{dx}{dt}(t) + f_y(x(t), y(t)) \frac{dy}{dt}(t).$$
Example of Chain rule, case 1

\( f(x, y) = x^2 + 2y^3, \quad x(t) = \sin(t), \quad y(t) = \cos(2t). \)

Let \( z(t) = f(x(t), y(t)). \) Then,

\[
\frac{dz}{dt} = 2x(t) \frac{dx}{dt} + 6[y(t)]^2 \frac{dy}{dt},
\]

\[
= 2x(t) \cos(t) - 12[y(t)]^2 \sin(2t),
\]

\[
= 2 \sin(t) \cos(t) - 12 \cos^2(2t) \sin(2t).
\]

Second case of chain rule for \( f(x, y) \)

Case 2: Let \( f(x, y) \) be differentiable, and \( x(t, s), y(t, s) \) be also differentiable functions of a two variable. Then \( z(t, s) = f(x(t, s), y(t, s)) \) is differentiable and

\[
z_t(t, s) = f_x(x(t, s), y(t, s)) x_t(t, s) + f_y(x(t, s), y(t, s)) y_t(t, s)
\]

\[
z_s(t, s) = f_x(x(t, s), y(t, s)) x_s(t, s) + f_y(x(t, s), y(t, s)) y_s(t, s)
\]
Second case of chain rule for $f(x, y)$ again

Case 2: Let $f(x, y)$ be differentiable, and $x(t, s), y(t, s)$ be also differentiable functions of a two variable. Then $z(t, s) = f(x(t, s), y(t, s))$ is differentiable and

$$z_t = f_x x_t + f_y y_t$$

$$z_s = f_x x_s + f_y y_s$$

Example of chain rule, case 2

A change of coordinates:

Consider the function $f(x, y) = x^2 + ay^2$, with $a \in \mathbb{R}$. Introduce polar coordinates $r, \theta$ by the formula

$$x(r, \theta) = r \cos(\theta), \quad y(r, \theta) = r \sin(\theta).$$

Let $z(r, \theta) = f(x(r, \theta), y(r, \theta))$ be $f$ in polar coordinates.
A change of coordinates

Then, the chain rule, case 2, says that

$$z_r = f_x x_r + f_y y_r.$$  

Each term can be computed as follows,

$$f_x = 2x, \quad f_y = 2ay,$$

$$x_r = \cos(\theta), \quad y_r = \sin(\theta),$$

then one has

$$z_r = 2r \cos^2(\theta) + 2ar \sin^2(\theta).$$