Criterias for convergence of series

- Review: A rough criteria.
- Integral test for convergence.
- Estimating the remainder.
- Comparison test for convergence.

Here is a very rough criteria to check if a series diverges

**Theorem 1** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

It is based in the following result:

**Theorem 2** If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0.$
Here is a test of convergence for certain series using integration

**Theorem 3** Let \( f(x) \) be a continuous, positive, decreasing function on \([1, \infty)\). Consider the sequence \( a_n = f(n), \ n \in \mathbb{Z}^+ \). Then,

\[ \sum_{n=1}^{\infty} a_n \ \text{is convergent} \iff \int_{1}^{\infty} f(x) \, dx \ \text{is convergent}. \]

The proof is to show the following inequality

\[ \int_{1}^{n+1} f(x) \, dx \leq s_n \leq a_1 + \int_{1}^{n} f(x) \, dx. \]

We already used this test in the harmonic series

The harmonic series \( s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \) diverges, because

\[ \ln(n+1) = \int_{1}^{n+1} \frac{1}{x} \, dx \leq s_n, \]

and \( \ln(n+1) \) diverges when \( n \to \infty \).
This test can be applied to the geometric series

Consider the geometric series

\[ s_n = \sum_{k=1}^{n} r^{(k-1)} = 1 + r + r^2 + r^3 + \ldots + r^{n-1}, \]

for \(0 < r < 1\).

Here \( f(x) = r^x \). Recall \( \int r^x \, dx = \frac{r^x}{\ln(r)} + c \). Then,

\[ s_n \leq 1 + \int_1^n r^x \, dx = 1 + \frac{r}{\ln(1/r)} - \frac{r^n}{\ln(1/r)}. \]

and \( r^n \to 0 \) when \( n \to \infty \). So, \( s_n \) converges.

We knew that the geometric series converges, \( \sum_{k=1}^{\infty} r^{(k-1)} = \frac{1}{1-r} \).

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This test can be applied to the \( p \)-series

**Theorem 4** The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges for \( p > 1 \), and it diverges for \( p \leq 1 \).

\[
\int_1^{\infty} \frac{1}{x^p} \, dx = \begin{cases} 
\frac{1}{p-1} & p > 1 \\
\text{diverges} & p \leq 1
\end{cases}
\]
The remainder of a series can be estimated with the same type of integral bounds as above.

**Definition 1** If \( s_n = \sum_{k=1}^{n} a_k \) converges to \( s \), then the remainder after \( n \) terms is

\[ R_n = s - s_n. \]

So, \( R_n = a_{n+1} + a_{n+2} + \cdots \), that is,

\[ R_n = \sum_{k=n+1}^{\infty} a_k. \]

Here is the estimation for the remainder.

**Theorem 5** Let \( f(x) \) be continuous, positive, decreasing function on \([n, \infty)\). Consider the sequence \( a_k = f(k) \) for \( k \geq n \). Suppose that \( \sum_{k=1}^{\infty} a_k \) is convergent. Then,

\[ \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx. \]

Therefore,

\[ s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s_n + \int_{n}^{\infty} f(x) \, dx. \]
There are three types of comparison tests

**Theorem 6** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.

1. If $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. If $\lim_{n \to \infty} a_n/b_n = c > 0$, finite, then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge, or both diverge.

More and more and more convergence tests

- Review: Integral and test.
- Alternating series.
- Absolute convergence
- Root and ratio tests.

Why so many tests? Sometimes works one, sometimes another.
The integral test links series convergence with and improper integral convergence

\( f(x) \) continuous, positive, and decreasing function on \([1, \infty)\). Define the sequence \( a_n = f(n), \ n \in \mathbb{Z}^+ \).

\[ \sum_{n=1}^{\infty} a_n \text{ is convergent } \iff \int_1^{\infty} f(x) \, dx \text{ is convergent.} \]

The proof is to show the following inequality

\[ \int_1^{n+1} f(x) \, dx \leq s_n \leq a_1 + \int_1^{n} f(x) \, dx. \]

The integral test does not apply on every series

**Definition 2** Given \( 0 < a_n \), the series

\[ s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \text{ is called alternating series.} \]

That is

\[ s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n. \]

Here is the alternating harmonic series:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]
The harmonic series diverges, however the alternating harmonic series converges

**Theorem 7** If the alternating series $\sum_{k=1}^{n}(-1)^{k+1}a_k$ with $a_k > 0$ also satisfies

- $a_{n+1} < a_n$ (decreasing sequence),
- $a_n \to 0$ as $n \to \infty$,

then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

The convergence of a nonpositive series could be determined by its absolute value series

**Definition 3** A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

Is the series below convergent?

$$s_n = \sum_{k=1}^{n} \frac{\cos(3k)}{k^2}$$

First notice that it is absolutely convergent.
Absolutely convergence implies convergence

**Theorem 8** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Therefore, the series
\[ s_n = \sum_{k=1}^{n} \frac{\cos(3k)}{k^2} \]
is indeed convergent.

The root test for a series is to compare it with a geometric series

**Theorem 9** Consider the series $\sum_{k=1}^{n} a_k$ and denote
\[ L = \lim_{n \to \infty} |a_n|^{1/n}. \]

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $L > 1$ or $L$ is $\infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $L = 1$, then the test is inconclusive.
The ratio test is perhaps the test one should try first.

**Theorem 10** Consider the series \( \sum_{k=1}^{n} a_k \) and denote

\[
L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}.
\]

- If \( L < 1 \), then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.
- If \( L > 1 \) or \( L = \infty \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.
- If \( L = 1 \), then the test is inconclusive.