$z = a + bi$, **powers, roots, and exponentials**

- Review: Cartesian and polar representations.
- Powers and roots.
- Exponential and Euler formula.

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**Slide 1**

**Slide 2**

Complex numbers can be associated with points in a plane

- Cartesian picture: Good for representing addition and real number multiplication.
  (Parallelogram law and stretching.)
- Polar picture: Good for representing the multiplication law.
  (Stretching and rotation.)
The power of a complex number is very easy to compute in the polar representation.

**Theorem 1 (De Moivre)**

\[(r[\cos(\theta) + i \sin(\theta)])^n = r^n[\cos(n\theta) + i \sin(n\theta)].\]

Equivalently:

\[z = r[\cos(\theta) + i \sin(\theta)] \Rightarrow z^n = r^n[\cos(n\theta) + i \sin(n\theta)].\]

Arbitrary powers are easy in polar representation.

\[z = a + bi, \quad \Leftrightarrow \quad z = r[\cos(\theta) + i \sin(\theta)],\]

\[r = \sqrt{a^2 + b^2}, \quad \theta = \arctan(b/a).\]

Then,

\[(a + bi)^n = r^n[\cos(n\theta) + i \sin(n\theta)].\]
Magic at work: There are $n$ solutions to the $n$-th root of a complex number

(In real numbers there are one or two, for $n$ is odd or even, respectively.)

**Theorem 2** Let $z = r[\cos(\theta) + i\sin(\theta)]$ and $n \geq 1$.

Then, the complex numbers

$$w_k = r^{\frac{1}{n}} \left[ \cos \left( \frac{\theta}{n} + \frac{2\pi}{n} k \right) + i \sin \left( \frac{\theta}{n} + \frac{2\pi}{n} k \right) \right]$$

$k = 0, \ldots, n - 1$ satisfy the equation

$$(w_k)^n = z.$$

**Why not to integrate by parts?**

- Review: Complex numbers and Euler formula.
- Integration by parts.
- Exercises.
- Recursion formula.
Euler first obtained a formula for the exponential of real numbers

**Theorem 3**

\[ e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n, \]

for all \( x \in \mathbb{R} \).

Euler later considered the De Moivre formula

\[ \left[\cos(\theta) + i \sin(\theta)\right] = \left[\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right]^n. \]

Therefore,

\[ \left[\cos(\theta) + i \sin(\theta)\right] = \lim_{n \to \infty} \left(1 + \frac{i\theta}{n}\right)^n. \]
Euler formula is one of the most beautiful formulas we have seen so far.

The calculation above suggests the following relation:

\[ e^{i\theta} = \cos(\theta) + i\sin(\theta). \]

In particular, one has Euler formula:

\[ e^{i\pi} - 1 = 0. \]

Why not to integrate by parts?

**Theorem 4 (Integration by parts)** If \( f(x) \) and \( g(x) \) are integrable functions in \([a, b]\), then the following formulas hold,

\[
\int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx,
\]

\[
\int_a^b f'(x)g(x) \, dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x) \, dx.
\]
The proof is based on the product rule and the FTC

Recall that \([f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)\).

Indefinite integral:

\[
\int [f(x)g(x)]' \, dx = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx.
\]

Definite integral:

\[
[f(x)g(x)]_a^b = \int_a^b [f(x)g(x)]' \, dx = \int_a^b f'(x)g(x) \, dx + \int_a^b f(x)g'(x) \, dx.
\]

Simple examples of integration by parts

Find the following integrals:

\[
\int \ln(x) \, dx = x \ln(x) - x,
\]

\[
\int xe^x \, dx = (x - 1)e^x,
\]

\[
\int x \sin(x) \, dx = -x \cos(x) + \sin(x),
\]

\[
\int \frac{1}{x} \ln(x) \, dx = \frac{1}{2} \ln^2(x).
\]
Integration by parts is very useful to construct integration tables

Do you know how the following integral was discovered?

\[ \int \frac{x^2}{2} e^x \, dx = \left( \frac{x^2}{2} - x + 1 \right) e^x. \]

Reduction formulas are a simple way to write complicated integrals

In the case of the function \( \sin(x) \) one has:

\[ \int [\sin(x)]^n \, dx = -\frac{1}{n} [\sin(x)]^{(n-1)} \cos(x) + \frac{(n-1)}{n} \int [\sin(x)]^{(n-2)} \, dx. \]