Slide 1

Distance formula

Theorem 1 *The distance between the points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *is given by*

$$|P_1P_2| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}.$$

The concept of distance has a central role to generalize the concept of limit to vector valued functions.

Application: A sphere has an equation.

$$S_{P_0,R} = \{P \in \mathbb{R}^3 : |P_0P| = R\},$$

is the sphere centered at $P_0(x_0, y_0, z_0)$ of radius $R > 0$. The equation is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

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Application: The equation of a ball centered at P_0 of radius R is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq R^2.$$

Exercises

- Fix constants a, b, c, and d. Show that

  $$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

  is the equation of a sphere if and only if

  $$d > -(a^2 + b^2 + c^2).$$

- Give the expressions for the center P_0 and the radius R of the sphere.

Slide 3

Vectors in $\mathbb{R}^3$

- What are vectors?
- Operations with vectors.
  - Addition, Difference.
  - Multiplication by a number.
- Components.

Slide 4

What are vectors?

$\sim$ 1800 Physicists and Mathematicians realized that several different physical phenomena were described using the same idea, the same concept. These phenomena included velocities, accelerations, forces, rotations, electric and magnetic phenomena, heat transfer, etc.

The new concept were more than a number in the sense that it was needed more than a single number to specify it.

Definition 1 *A vector in $\mathbb{R}^3$ is an oriented segment.*

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Operations with vectors

- Addition: Parallelogram law.
- Multiplication by a number. (Positive, negative, or zero.)
- Difference.

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Components on a vector

The operations with vectors, defined geometrically can be written in terms of components.

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the following expressions hold,

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle, \\ \mathbf{v} - \mathbf{w} &= \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle, \\ a\mathbf{v} &= \langle a v_x, a v_y, a v_z \rangle, \\ |\mathbf{v}| &= \left[(v_x)^2 + (v_y)^2 + (v_z)^2 \right]^{1/2}. \end{aligned}$$

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Useful vectors:

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle, \\ \mathbf{j} &= \langle 0, 1, 0 \rangle, \\ \mathbf{k} &= \langle 0, 0, 1 \rangle, \end{aligned}$$

Every vector $\mathbf{v}$ in $\mathbb{R}^3$ can be written uniquely in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

The following equation holds,

$$\mathbf{v} = \langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

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Dot Product

- Definition
- Properties
- Equivalent expression

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Definition and properties

Definition 2 *Let* $\mathbf{v}$, $\mathbf{w}$ *be vectors and* $0 \leq \theta \leq \pi$ *be the angle in between. Then*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta).$$

Properties:

- $\mathbf{v} \cdot \mathbf{w} = 0 \Longleftrightarrow \mathbf{v} \perp \mathbf{w}, \quad (\theta = \pi/2);$
- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2, \quad (\theta = 0);$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}, \qquad$ (commutative);
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$

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Equivalent expression

Theorem 2 *Let* $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$. *Then*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

For the proof, recall that

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\begin{array}{rclrclrcl} \mathbf{i} \cdot \mathbf{i} &=& 1, & \mathbf{j} \cdot \mathbf{j} &=& 1, & \mathbf{k} \cdot \mathbf{k} &=& 1, \\ \mathbf{i} \cdot \mathbf{j} &=& 0, & \mathbf{j} \cdot \mathbf{i} &=& 0, & \mathbf{k} \cdot \mathbf{i} &=& 0, \\ \mathbf{i} \cdot \mathbf{k} &=& 0, & \mathbf{j} \cdot \mathbf{k} &=& 0, & \mathbf{k} \cdot \mathbf{j} &=& 0 \end{array}$$

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Cross Product

- Definition
- Properties (Determinants)
- Equivalent expression
- Triple product

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Definition

Definition 3 *Let* $\mathbf{v}$, $\mathbf{w}$ *be 3-dimensional vectors, and* $0 \leq \theta \leq \pi$ *be the angle in between them. Then,* $\mathbf{v} \times \mathbf{w}$ *is a vector normal to* $\mathbf{v}$ *and* $\mathbf{w}$*, pointing in the direction given by the right hand rule, and with norm*

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin(\theta).$$

Example:

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

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Properties

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$,
- $\mathbf{v} \times \mathbf{v} = 0$,
- $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$,
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Notice: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Example: $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$, but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = 0$.

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Theorem 3 *If* $\mathbf{v}$, $\mathbf{w} \neq 0$, *then the following assertion holds:*

$$\mathbf{v} \times \mathbf{w} = 0 \Leftrightarrow \mathbf{v} \textit{ parallel } \mathbf{w}.$$

Theorem 4 $|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by* $\mathbf{v}$ *and* $\mathbf{w}$.

Theorem 5 *Let* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. *Then,*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

For the proof of the last theorem, recall that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

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Note on determinants

They are useful un several areas of Mathematics. We don't study them in our course. We use them only as a tool to remember the components of $\mathbf{v} \times \mathbf{w}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

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Triple product

Definition 4 *Given* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *the triple product is the number given by*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Theorem 6 *Fix nonzero vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. *Then,* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Note: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$.