Determinants (Sec. 3.2)

- Review: Definition of determinant of $n \times n$ matrices.
- Properties of determinants.
- Determinants and elementary row operations.
- Determinant of a product of matrices.

**Review: Definition of determinant**

**Definition 1** The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} \det(A_{1j}) a_{1j}.$$  

*This formula is called “expansion by the first row.”*
Properties

Theorem 1 (Main properties of $n \times n$ determinants) Let $A = [a_1, \ldots, a_n]$ be an $n \times n$ matrix. Let $c$ be an $n$-vector.

- $\det([a_1, \ldots, a_j + c, \ldots, a_n]) = \det([a_1, \ldots, a_j, \ldots, a_n]) + \det([a_1, \ldots, c, \ldots, a_n])$.

- $\det([a_1, \ldots, ca_j, \ldots, a_n]) = c \det([a_1, \ldots, a_j, \ldots, a_n])$.

- $\det([a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n]) = -\det([a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n])$.

- $\det([a_1, \ldots, a_i, \ldots, a_i, \ldots, a_n]) = 0$.

- $\det(A) = \det(A^T)$.

- \{a_1, \ldots, a_n\} are l.d. $\Leftrightarrow \det([a_1, \ldots, a_n]) = 0$.

- A is invertible $\Leftrightarrow \det(A) \neq 0$.

Properties

The properties of the determinant on the column vectors of $A$ and the property $\det(A) = \det(A^T)$ imply the following results on the rows of $A$.

Theorem 2 (Determinants and elementary row operations) Let $A$ be a $n \times n$ matrix.

- Let $B$ be the result of adding to a row in $A$ a multiple of another row in $A$. Then, $\det(B) = \det(A)$.

- Let $B$ be the result of interchanging two rows in $A$. Then, $\det(B) = -\det(A)$.

- Let $B$ be the result of multiply a row in $A$ by a number $k$. Then, $\det(B) = k \det(A)$. 
Determinant and elementary row operations

**Theorem 3** If $E$ represents an elementary row operation and $A$ is an $n \times n$ matrix, then

\[
\det(EA) = \det(E) \det(A).
\]

The proof is to compute the determinant of every elementary row operation matrix, $E$, and then use the previous theorem.

**Theorem 4 (Determinant of a product)** If $A$, $B$ are arbitrary $n \times n$ matrices, then

\[
\det(AB) = \det(A) \det(B).
\]

**Determinant of a product of matrices**

*Proof:* If $A$ is not invertible, then $AB$ is not invertible, then the theorem holds, because $0 = \det(AB) = \det(A) \det(B) = 0$. Suppose that $A$ is invertible. Then there exist elementary row operations $E_k, \ldots, E_1$ such that

\[
A = E_k \cdots E_1.
\]

Then,

\[
\begin{align*}
\det(AB) &= \det(E_k \cdots E_1 B), \\
&= \det(E_k) \det(E_{k-1} \cdots E_1 B), \\
&= \det(E_k) \cdots \det(E_1) \det(B), \\
&= \det(E_k \cdots E_1) \det(B), \\
&= \det(A) \det(B).
\end{align*}
\]
Formula for the inverse matrix

- Formula for the inverse matrix.
- Application to systems of linear equations.

Theorem 5 Let $A$ be an $n \times n$ matrix with components $(A)_{ij} = a_{ij}$. Let $C_{ij} = (-1)^{i+j} \det(A_{ij})$ be the $ij$th cofactor, and $\Delta = \det(A)$. Then the component $ij$ of the inverse matrix $A^{-1}$ is given by

$$(A^{-1})_{ij} = \frac{1}{\Delta} [C_{ji}].$$

That is,

$$A^{-1} = \frac{1}{\Delta} 
\begin{bmatrix}
  C_{11} & C_{21} & \cdots & C_{n1} \\
  C_{12} & C_{22} & \cdots & C_{n2} \\
  \vdots & \vdots & \ddots & \vdots \\
  C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}.$$
Formula for the inverse matrix

Proof: It is a straightforward computation. Let us denote $B$ the matrix with components $(B)_{ij} = C_{ji}/\Delta$. Then,

$$
AB = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}
\frac{1}{\Delta}
$$

Compute each component of the product $AB$.

$$(AB)_{11} = \frac{1}{\Delta}(C_{11}a_{11} + C_{12}a_{12} + \cdots + C_{1n}a_{1n}) = 1,$$

because the factor in the numerator in the right hand side is precisely $\det(A) = \Delta$.

The second component is given by

$$(AB)_{12} = \frac{1}{\Delta}(C_{11}a_{21} + C_{12}a_{22} + \cdots + C_{1n}a_{2n}).$$

The factor between brackets in the right hand side is an expansion by the first row of the determinant of a matrix whose first row is $a_{21}, a_{22}, \cdots a_{2n}$.

That is,

$$
(AB)_{12} = \frac{1}{\Delta}
\begin{vmatrix}
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
= 0.
$$
An analogous calculation shows that \((AB)_{ij}\) is given by

\[
(AB)_{ij} = \frac{1}{\Delta} (C_{j1}a_{i1} + C_{j2}a_{i2} + \cdots + C_{jn}a_{in}),
\]

The factor between brackets in the right hand side is an expansion by the \(j\) row of the determinant of a matrix whose \(j\) row is the \(i\) row of \(A\),

\[
a_{i1}, a_{i2}, \ldots, a_{in}.
\]

That is,

\[
(AB)_{ij} = \frac{1}{\Delta} \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]
in the \(j\)-row

Therefore, when \(i \neq j\) the factor between brackets is the determinant of a matrix with two identical rows, so \((AB)_{ij} = 0\) for \(i \neq j\). If \(i = j\), the that factor is precisely \(\det(A)\), then \((AB)_{ii} = 1\).

Summarizing,

\[
(AB)_{ij} = \frac{1}{\Delta} \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
\]
in the \(j\)-row

\[= I_{ij}\]

Repeat this calculation for \(BA\).
Systems of linear equations

**Theorem 6** Suppose that the matrix $A$ is invertible. Then the system of linear equations $Ax = b$ has a unique solution for every vector $b$. If $x_i$ are the components of $x$ and $A_i(b) = [a_1, \cdots, b, \cdots, a_n]$, with $b$ in the $i$ column, then

$$x_i = \frac{1}{\Delta} \det(A_i(b)).$$

**Proof:** $A$ invertible means that the solution can be written as $x = A^{-1}b$. From the formula of the inverse matrix one has that

$$x_i = \frac{1}{\Delta}(C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n),$$

where $b_i$ are the components of $b$. Notice that if one expands the $\det(A_i(b))$ by the $i$ row one gets

$$\det(A_i(b)) = (C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n).$$