Components and change of basis

- Review: Isomorphism.
- Review: Components in a basis.
- Unique representation in a basis.
- Change of basis.

Review: Isomorphism

**Definition 1 (Isomorphism)** The linear transformation $T : V \rightarrow W$ is an isomorphism if $T$ is one-to-one and onto.

Example: $T : P_1 \rightarrow \mathbb{R}^3$ given by

$$T(a + bt + ct^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is an isomorphism.
Basis and components

Definition 2 (Dimension) A vector space $V$ has dimension $n$ if the maximum number of l.i. vectors is $n$.

Definition 3 (Basis) A basis of an $n$-dimensional vector space $V$ is any set $\{u_1, \ldots, u_n\}$ of $n$ l.i. vectors.

Theorem 1 (Basis) Let $V$ be an $n$-dimensional vector space. The set $\{u_1, \ldots, u_n\}$ is a basis of $V \iff \{u_1, \ldots, u_n\}$ are l.i. and they span $V$.

Theorem 2 Let $\{u_1, \ldots, u_n\}$ be a basis of $V$. Then, each vector $v \in V$ has a unique decomposition

$$v = a_1 u_1 + \cdots + a_n u_n.$$  

Proof of Theorem 1:

($\Rightarrow$) Suppose that $\{u_1, \ldots, u_n\}$ does not span $V$. Then there exists $v \in V$ that it is not a linear combination of $\{u_1, \ldots, u_n\}$. That is, $bv + a_1 u_1 + \cdots + a_n u_n = 0$ implies that $b = a_1 = \cdots = a_n = 0$. This in turn says that $\{v, u_1, \ldots, u_n\}$ is a l.i. set. But this is a contradiction with the assumption that $n$ is the maximum number of l.i. vectors in $V$.

($\Leftarrow$) $\{u_1, \ldots, u_n\}$ is l.i. and spans $V$. Then, for all $v \in V$ there exists numbers $a_1, \ldots, a_n$ such that $v = a_1 u_1 + \cdots + a_n u_n$. That is, the set $\{v, u_1, \ldots, u_n\}$ is l.d. for all $v \in V$. That says that $n$ is the maximum number of l.i. vectors in $V$.  

Proof of Theorem 2: The set $\{u_1, \ldots, u_n\}$ is a basis of $V$ so they span $V$. Then, there exists scalars $a_i$, for $1 \leq i \leq n$ such that the following decomposition holds,

$$v = \sum_{i=1}^{n} a_i u_i.$$  

This decomposition is unique. Because, if there is another decomposition

$$v = \sum_{i=1}^{n} b_i u_i.$$  

then the difference has the form

$$\sum_{i=1}^{n} (a_i - b_i) u_i = 0.$$  

Because the vectors $\{u_1, \ldots, u_n\}$ are l.i. this implies that $a_i = b_i$ for all $0 \leq i \leq n$.  

Exercises

Consider the basis \( \{ e_1, e_2 \} \) given by
\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Consider a second basis \( \{ u_1, u_2 \} \) given by
\[
u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Find the components of \( x = e_1 + 2e_2 \) in the basis \( \{ u_1, u_2 \} \).

\[
x = e_1 + 2e_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [x]_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

The vectors \( \{ u_1, u_2 \} \) form a basis so there exists constants \( c_1, c_2 \) such that
\[
x = c_1 u_1 + c_2 u_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad [x]_u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\]

Therefore,
\[
[x]_e = c_1 [u_1]_e + c_2 [u_2]_e.
\]

That is,
\[
\begin{bmatrix} 1 \\ 2 \end{bmatrix}_e = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u
\]

Then one has to solve the augmented matrix
\[
\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix},
\]

so \( c_1 = 3/2 \) and \( c_2 = -1/2 \), and then
\[
[x]_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [x]_u = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}.
\]
Change of basis

Theorem 3 (Change of basis) Let \(\{u_1, \ldots, u_n\}\) and \(\{v_1, \ldots, v_n\}\) be basis of \(V\). Then, there exists a unique \(n \times n\) invertible matrix \(P_{v \leftrightarrow u}\) such that

\[
[x]_v = P_{v \leftrightarrow u}[x]_u,
\]

for all \(x \in V\). Furthermore, the matrix \(P_{v \leftrightarrow u}\) has the form

\[
P_{v \leftrightarrow u} = [[u_1]_v, \cdots, [u_n]_v].
\]

Proof of Theorem 3: Both sets \(\{u_1, \ldots, u_n\}\) and \(\{v_1, \ldots, v_n\}\) are basis of \(V\), then there exist a unique set of numbers \(\{u_1, \ldots, u_n\}\) and \(\{v_1, \ldots, v_n\}\) such that

\[
x = u_1 u_1 + \cdots + u_n u_n = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad x = v_1 v_1 + \cdots + v_n v_n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.
\]

Therefore,

\[
\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = [[u_1]_v, \cdots, [u_n]_v] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.
\]

This system of equations for \((u_1, \ldots, u_n)\) has a unique solution solutions for all \((v_1, \ldots, v_n)\), because the \(u\)'s and \(v\)'s are basis. That is, \(P_{v \leftrightarrow u} = [[u_1]_v, \cdots, [u_n]_v]\) is invertible. \(\blacksquare\)
Exercises

• (2, Sec. 4.7) Let \( \{b_1, b_2\}, \{c_1, c_2\} \) be basis of \( \mathbb{R}^2 \). Let 
  \( b_1 = -c_1 + 4c_2 \) and \( b_2 = 5c_1 - 3c_2 \).
  
  - Find \([x]_c\) for \([x]_b = (5, 3)_b\).
  - Find \([x]_b\) for \([x]_c = (1, 1)_c\).

In \( P_2 \) find the change of coordinate matrix from the basis 
\( B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + t^2\} \) to the standard basis 
\( \{1, t, t^2\} \). Find the \( B \)-coordinates of \( x = 1 - 2t \).

Determinants

• Determinants of \( 2 \times 2 \) matrices.
• Definition.
• Properties.
Slide 8

$2 \times 2$ determinant

Definition 4 The determinant of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$  

The determinant appears in the computation of the inverse matrix.

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Slide 9

Properties

Theorem 4 (Main properties of $2 \times 2$ determinants)

Let $A = [a_1, a_2]$ be a $2 \times 2$ matrix. Let $c$ be a 2-vector.

- $\det([a_1 + c, a_2]) = \det([a_1, a_2]) + \det([c, a_2])$.
- $\det([ca_1, a_2]) = c \det([a_1, a_2])$.
- $\det([a_1, a_2]) = -\det([a_2, a_1])$.
- $\det([a_1, a_1]) = 0$.
- $a_1, a_2$ are l.d. $\iff \det([a_1, a_2]) = 0$.
- $A$ is invertible $\iff \det(A) \neq 0$.
- $\det(A) = \det(A^T)$. 


Properties

Theorem 5 (Determinants and elementary row operations)
Let $A$ be a $2 \times 2$ matrix.

- Let $B$ be the result of adding to a row in $A$ a multiple of another row in $A$. Then, $\det(B) = \det(A)$.
- Let $B$ be the result of interchanging two rows in $A$. Then, $\det(B) = -\det(A)$.
- Let $B$ be the result of multiplying a row in $A$ by a number $k$. Then, $\det(B) = k \det(A)$.

Determinants and areas

Theorem 6 Let $A = [a_1, a_2]$ be a $2 \times 2$ matrix, with $a_1$ and $a_2$ being nonzero and noncollinear. Then, $|\det([a_1, a_2])|$ is the area of the parallelogram formed by $a_1$ and $a_2$.

Proof: Choose a basis $e_1, e_2$ such that $a_1 = be_1$, for some number $b \neq 0$. Because $a_1$ is not collinear to $a_2$, there exists a $c \neq 0$ such that the vector $u = ca_1 + a_2$ is collinear to $e_2$. For that vector $u$ holds that $u = he_2$, where $|h|$ is the height of the parallelogram. Summarizing:

$$a_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Now, the determinant of $A$ is the following:

$$\det(A) = \det([a_1, a_2]) = \det([a_1, a_2 + ca_1]) = \det([a_1, u]).$$
Therefore,
\[ \det(A) = \begin{vmatrix} b & 0 \\ 0 & h \end{vmatrix} = hb. \]

Then, \( |\det(A)| = |h||b| \), where \(|h|\) is the height and \(|b|\) the length of the base of the parallelogram.
**Determinant of 3 x 3 matrices**

**Definition 5** The determinant of a 3 x 3 matrix $A$ is given by

$$
\det(A) = \begin{vmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{vmatrix}
$$

$$= a_{22} a_{33} a_{11} - a_{32} a_{23} a_{11} - a_{21} a_{33} a_{12} + a_{31} a_{23} a_{12} + a_{21} a_{32} a_{13} - a_{31} a_{22} a_{13},
$$

This formula is called “expansion by the first row.”

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**Theorem 7 (Expansions by rows)** The determinant of a 3 x 3 matrix $A$ can also be computed with an expansion by the second row or by the third row.

The proof is just do the calculation. For example, the expansion by the second row is the following:

$$
- a_{12} a_{13} a_{21} + a_{11} a_{13} a_{22} - a_{11} a_{12} a_{23} = -(a_{12} a_{33} - a_{32} a_{13}) a_{21} + (a_{11} a_{33} - a_{31} a_{13}) a_{22} - (a_{11} a_{32} - a_{31} a_{12}) a_{23} = \det(A).
$$
**Determinant of 3 × 3 matrices**

**Theorem 8 (Expansions by columns)** The determinant of a 3 × 3 matrix $A$ can also be computed with an expansion by any of its columns.

The proof is again to do the calculation. For example, the expansion by the first column is the following:

$$
\begin{vmatrix}
    a_{22} & a_{23} & a_{11} - a_{21} & a_{12} & a_{13} \\
    a_{32} & a_{33} & a_{21} + a_{12} & a_{23} & a_{31}
\end{vmatrix}
$$

$$
= (a_{22}a_{33} - a_{23}a_{32})a_{11} - (a_{12}a_{33} - a_{32}a_{13})a_{21} + (a_{12}a_{23} - a_{22}a_{13})a_{31}
= \det(A).
$$

**Determinant of n × n matrices**

Notation:

$$
A_{ij} = 
\begin{bmatrix}
    a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
    \vdots & & \vdots & & \vdots \\
    a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
    \vdots & & \vdots & & \vdots \\
    a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
$$

$$
A = 
\begin{bmatrix}
    + & - & + & \cdots \\
    - & + & - & \cdots \\
    + & - & + & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \text{ Sign of coefficient } a_{ij} \text{ is } (-1)^{i+j}.
Determinant of $n \times n$ matrices

**Definition 6** The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$
\det(A) = \det(A_{11})a_{11} - \det(A_{12})a_{12} + \cdots + (-1)^{1+n} \det(A_{1n})a_{1n},
$$

$$
= \sum_{j=1}^{n} (-1)^{1+j} \det(A_{1j})a_{1j}.
$$

This formula is called “expansion by the first row.”

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**Theorem 9** The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be computed by an expansion along any row or along any column. That is,

$$
\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij})a_{ij}, \quad \text{for any } i = 1, \ldots, n,
$$

$$
= \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij})a_{ij}, \quad \text{for any } j = 1, \ldots, n.
$$

Notation: The cofactor $C_{ij}$ of a matrix $A$ is the number given by

$$
C_{ij} = (-1)^{i+j} \det(A_{ij}).
$$

**Theorem 10** $\det(A) = \det(A^T)$. 
**Determinant of \( n \times n \) matrices**

Suggestion: If a matrix has a row or a column with several zeros, then it is simpler to compute its determinant by an expansion along that row or column.

**Theorem 11** The determinant of a triangular matrix is the product of its diagonal elements.

Examples:

\[
\begin{vmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{vmatrix} = \begin{vmatrix}
4 & 5 \\
0 & 6
\end{vmatrix} = 4 \times 5 = 20.
\]

\[(1) = 1 \times 4 \times 6 = 24.\]

\[
\begin{vmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
4 & 1 & 5
\end{vmatrix} = \begin{vmatrix}
3 & 0 \\
1 & 5
\end{vmatrix} = 3 \times 5 = 15.
\]

\[(1) = 1 \times 3 \times 5 = 15.\]