Cylindrical and spherical coordinates

- Review of Polar coordinates in \( \mathbb{R}^2 \).
- Cylindrical coordinates in \( \mathbb{R}^3 \).
- Spherical coordinates in \( \mathbb{R}^3 \).
- Exercises.

Polar coordinates in \( \mathbb{R}^2 \)

Definition 1 (Polar coordinates) Let \((x,y)\) be Cartesian coordinates in \( \mathbb{R}^2 \). Then, polar coordinates \((r, \theta)\) are defined in \( \mathbb{R}^2 - \{(0,0)\} \), and given by

\[
  r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).
\]

The inverse expression is

\[
  x = r \cos(\theta), \quad y = r \sin(\theta).
\]
Definitions

Definition 2 (Cylindrical coordinates) Let \((x, y, z)\) be Cartesian coordinates in \(\mathbb{R}^3\). Then, cylindrical coordinates \((r, \theta, z)\) are defined in \(\mathbb{R}^3 - \{(0, 0, z)\}\), and given by

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z. \\
\end{align*}
\]

The inverse expression is

\[
\begin{align*}
    x &= r \cos(\theta), \quad y = r \sin(\theta), \quad z = z. \\
\end{align*}
\]

Computing Riemann sums in cylindrical coordinates one obtain the following formula for triple integrals,

\[
\int \int \int_I f \, dV = \int \int \int_I f(x, y, z) \, rdr \, d\theta \, dz. 
\]

Slide 3

Definitions

Definition 3 (Spherical coordinates) Let \((x, y, z)\) be Cartesian coordinates in \(\mathbb{R}^3\). Then, spherical coordinates \((r, \theta, \phi)\) are defined in \(\mathbb{R}^3 - \{(0, 0, z)\}\), and given by

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right). \\
\end{align*}
\]

The inverse expression is

\[
\begin{align*}
    x &= r \sin(\phi) \cos(\theta), \quad y = r \sin(\phi) \sin(\theta), \quad z = r \cos(\phi). \\
\end{align*}
\]

Computing Riemann sums in spherical coordinates one obtain the following formula for triple integrals,

\[
\int \int \int_I f \, dV = \int \int \int_I f(x, y, z) \, r^2 \sin(\phi) \, dr \, d\theta \, d\phi. 
\]

Slide 4
Exercises

- Find the volume of a cylinder of radius $r_0$ and height $h_0$.
  (Answer: $V = \pi r_0^2 h_0$.)
- Find the volume of a cone of base radius $r_0$ and height $h_0$.
  (Answer: $V = \pi r_0^2 h_0/3$.)
- (Sec. 15.8, Probl. 2) Find the solid whose volume is given by
  \[ V = \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta. \]

Exercises

- Find the volume of a sphere of radius $r_0$.
  (Answer: $V = (4/3)\pi r_0^3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi (2 - \sqrt{2})/3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi/8$.)
Spherical coordinates

- Review: Definition of spherical coordinates.
- Exercises.

Definitions

Definition 4 (Spherical coordinates) Let \((x, y, z)\) be Cartesian coordinates in \(\mathbb{R}^3\). Then, spherical coordinates \((r, \theta, \phi)\) are defined as follows:

\[
r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan \left( \frac{y}{x} \right), \quad \phi = \arctan \left( \frac{\sqrt{x^2 + y^2}}{z} \right),
\]

with the inverse expression given by

\[
x = r \sin(\phi) \cos(\theta), \quad y = r \sin(\phi) \sin(\theta), \quad z = r \cos(\phi).
\]

where \(0 < r\), \(0 < \phi < \pi\), and \(0 \leq \theta < 2\pi\).

Computing Riemann sums in spherical coordinates one obtain the following formula for triple integrals,

\[
\int \int \int_R f \, dV = \int \int \int_R f(x, y, z) \, r^2 \sin(\phi) \, dr \, d\phi \, d\theta.
\]
Exercises

- Find the volume of a sphere of radius $r_0$.
  (Answer: $V = (4/3)\pi r_0^3$.)

- Find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi(2 - \sqrt{2})/3$.)

- Find the volume below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$.
  (Answer: $V = \pi/8$.)

Exercises

- (Probl. 20, Sec. 15.8) Compute the integral
  \[ I = \int \int \int_R e^{\sqrt{x^2 + y^2 + z^2}} dV, \]
  where the region $R$ is the portion within the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.
  (Answer: $I = \pi(5e^3 - 1)/2$.)

- (Probl. 35, Sec. 15.8) Change to spherical coordinates and compute the following integral,
  \[ I = \int_{-\pi}^{\pi} \int_{-\sqrt{9-\rho^2}}^{\sqrt{9-\rho^2}} \int_{0}^{\sqrt{9-\rho^2}} x \sqrt{x^2 + y^2 + z^2} dz dy dx. \]
  (Answer: $I = 3^5 \pi/5$.)