Double integrals (Sec. 15.1 - 15.2)

- Review of the integral of single variable functions.
- Definition of double integral on rectangles.
- Average of a function.
- Double integrals general domains (Sec. 15.2).
- Examples of double integrals.

Integral of a single variable function

Definition 1 (Integral of single variable functions) Let \( f(x) \) be a function defined on an interval \( x \in [a,b] \). The integral of \( f(x) \) in \([a,b]\) is the number given by

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i^*) \Delta x,
\]

if the limit exists. Given a natural number \( n \) we have introduced a partition on \([a,b]\) given by \( \Delta x = (b-a)/n \). We denoted \( x_i^* = (x_i + x_{i-1})/2 \), where \( x_i = a + i\Delta x \), \( i = 0, 1, \ldots, n \). This choice of the sample point \( x_i^* \) is called midpoint rule.
Double integrals on rectangles

Definition 2 (Double integrals on rectangles) Let $f(x, y)$ be a function defined on a rectangle $R = [x_0, x_1] \times [y_0, y_1]$. The integral of $f(x, y)$ in $R$ is the number given by

$$\int \int_R f(x) \, dx \, dy = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y,$$

if the limit exists. Given a natural number $n$ we have introduced a partition on $R$ by rectangles of side $\Delta x = (x_1 - x_0)/n$, $\Delta y = (y_1 - y_0)/n$. We denoted $x_i^* = (x_i + x_{i-1})/2$, $y_j^* = (y_j + y_{j-1})/2$, where $x_i = x_0 + i \Delta x$, and $y_j = y_0 + j \Delta y$, for $i, j = 0 \cdots, n$. This choice of the sample point $x_i^*, y_j^*$ is called midpoint rule.

Notice: If $f(x, y) \geq 0$, then $\int \int_R f(x, y) \, dx \, dy = V$ the volume above $R$ and below the surface given by the graph of $f(x, y)$.

Read example 3, Sec. 5.1.
**Average**

The average value of a single variable function $f(x)$ is a number $\overline{f}$ such that the area below the graph of $f$ in the interval $[a, b]$ is given by:

$$A = (b - a)\overline{f}.$$  

Therefore, one has the formula:

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$  

**Definition 3 (Average)** The average of a function $f(x, y)$ in the domain $R = [x_0, x_1] \times [y_0, y_1]$ is denoted by $\overline{f}$, and it is given by the expression

$$\overline{f} = \frac{1}{A(R)} \iint_R f(x, y) \, dx\,dy,$$

with $A(R) = (x_1 - x_0)(y_1 - y_0)$ the area of the rectangle domain $R$.

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**Double integrals**

**Theorem 1 (Fubini)** If $f(x, y)$ is a continuous function in $R = [x_0, x_1] \times [y_0, y_1]$, then

$$\iint_R f(x, y) \, dxdy = \int_{y_0}^{y_1} \left[ \int_{x_0}^{x_1} f(x, y) \, dx \right] dy, \quad \text{or} \quad \iint_R f(x, y) \, dxdy = \int_{x_0}^{x_1} \left[ \int_{y_0}^{y_1} f(x, y) \, dy \right] dx.$$

Notation: One also denotes the double integral as

$$\iint_R f(x, y) \, dxdy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) \, dx\,dy.$$
Examples

\[ \int_1^3 \int_0^2 (xy^2 + 2x^2y^3) \, dxdy = \int_1^3 \left[ \int_0^2 (xy^2 + 2x^2y^3) \, dx \right] \, dy \]
\[ = \int_1^3 \left[ \frac{1}{2} y^2 x^2 \Big|_0^2 + \frac{2}{3} y^3 x^3 \Big|_0^2 \right] \, dy \]
\[ = \int_1^3 \left[ 2y^2 + \frac{16}{3} y^3 \right] \, dy, \]
\[ = \frac{2}{3} y^3 \Big|_1^3 + \frac{16}{12} y^4 \Big|_1^3, \]
\[ = \frac{2}{3} 26 + \frac{4}{3} 80. \]

Examples

\[ \int_1^4 \int_1^2 \left( \frac{x}{y} + \frac{y}{x} \right) \, dydx = \int_1^4 \left[ \int_1^2 \left( \frac{x}{y} + \frac{y}{x} \right) \, dy \right] \, dx, \]
\[ = \int_1^4 \left[ x \ln(y) \Big|_1^2 + \frac{1}{2x} \left( y^2 \right)_1^2 \right] \, dx, \]
\[ = \int_1^4 \left[ \ln(2)x + \frac{3}{2x} \right] \, dx, \]
\[ = \ln(2) \frac{1}{2} x^2 \Big|_1^4 + \frac{3}{2} \ln(x) \Big|_1^4, \]
\[ = \frac{15}{2} \ln(2) + \frac{3}{2} \ln(4), \]
\[ = \left( \frac{15}{2} + 3 \right) \ln(2). \]
Notice:

Fubini theorem, in the case of \( f(x, y) = g(x)h(y) \) says that:

\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} g(x)h(y)dydx = \left( \int_{x_0}^{x_1} g(x)dx \right) \left( \int_{y_0}^{y_1} h(y)dy \right).
\]

Example:

\[
\int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} dydx = \left[ \int_0^2 (1 + x^2)dx \right] \left[ \int_0^1 \frac{1}{1 + y^2} dy \right],
\]

\[
= \left( x|_0^2 + \frac{1}{3} x|_0^2 \right) \left( \arctan(y)|_0^1 \right),
\]

\[
= \frac{\pi}{4} \left( 2 + \frac{8}{3} \right).
\]

**Double integrals on regions (Sec. 15.3)**

- Regions function of \( y \).
- Regions function of \( x \).
- Properties of double integrals.
Regions functions of $y$

**Theorem 2 (Type I)** Let $g_0(x)$, $g_1(x)$ be two continuous functions defined on an interval $[x_0, x_1]$, and such that $g_0(x) \leq g_1(x)$. Let $f(x, y)$ be a continuous function in

$$D = \{(x, y) \in \mathbb{R}^2 : x_0 \leq x \leq x_1, \ g_0(x) \leq y \leq g_1(x)\}.$$ 

Then, the integral of $f(x, y)$ in $D$ is given by

$$\int \int_D f(x, y) \, dx \, dy = \int_{x_0}^{x_1} \left[ \int_{g_0(x)}^{g_1(x)} f(x, y) \, dy \right] \, dx.$$ 

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**Example: Type I**

- Find the $\int \int_D f(x, y) \, dx \, dy$ for

  $$f(x, y) = x^2 + y^2,$$

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ x^2 \leq y \leq x\}.$$
\[
\int \int_D f(x,y) \, dx \, dy = \int_0^1 \left[ \int_{x^2}^x (x^2 + y^2) \, dy \right] \, dx,
\]
\[
= \int_0^1 \left[ x^2 \left( y_{1/2}^2 \right) + \frac{1}{3} (y_{3/2}^2) \right] \, dx,
\]
\[
= \int_0^1 \left[ x^2 (x - x^2) + \frac{1}{3} (x^3 - x^6) \right] \, dx,
\]
\[
= \int_0^1 \left[ x^3 - x^4 + \frac{1}{3} x^3 - \frac{1}{3} x^6 \right] \, dx,
\]
\[
= \frac{1}{4} x^4|_0^1 - \frac{1}{5} x^5|_0^1 + \frac{1}{12} x^4|_0^1 - \frac{1}{21} x^7|_0^1,
\]
\[
= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{3 \times 5 \times 7}.
\]

\textbf{Regions functions of } x

\textbf{Theorem 3 (Type II)} Let \( h_0(y) \), \( h_1(y) \) be two continuous functions defined on an interval \([y_0, y_1]\), and such that \( h_0(y) \leq h_1(y) \). Let \( f(x,y) \) be a continuous function in
\[
D = \{(x,y) \in \mathbb{R}^2 : h_0(y) \leq x \leq h_1(y), \quad y_0 \leq y \leq y_1\}.
\]
Then, the integral of \( f(x,y) \) in \( D \) is given by
\[
\int \int_D f(x,y) \, dx \, dy = \int_{y_0}^{y_1} \left[ \int_{h_0(y)}^{h_1(y)} f(x,y) \, dx \right] \, dy.
\]
Example type II

- Find the $\int \int_D f(x, y) \, dx \, dy$ for

$$f(x, y) = x^2 + y^2,$$

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}.$$

Notice that $h_0(y) = y$, and $h_1(y) = \sqrt{y}$. Then,

$$D = \{(x, y) \in \mathbb{R}^2 : h_0(y) = y \leq x \leq h_1(y) = \sqrt{y}, \quad y_0 \leq y \leq y_1\}.$$

$$\int \int_D f(x, y) \, dx \, dy = \int_0^1 \left[ \int_y^{\sqrt{y}} (x^2 + y^2) \, dx \right] \, dy,$$

$$= \int_0^1 \left[ \frac{1}{3} (x^3 \sqrt{y}) + y^2 (x \sqrt{y}) \right] \, dy,$$

$$= \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] \, dy,$$

$$= \int_0^1 \left[ \frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + y^{5/2} - y^3 \right] \, dy,$$

$$= \frac{1}{3} \left[ \left. \frac{2}{5} y^{5/2} \right|_0^1 - \frac{1}{3} y^4 \right|_0^1 + \frac{2}{7} y^{7/2} \right|_0^1 - \frac{1}{4} y^4 \right|_0^1,$$

$$= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{3 \times 5 \times 7}.$$