Limits and Continuity

- Review of Limit.
- Side limits and squeeze theorem.
- Continuous functions of 2, 3 variables.

Review: Limits

Definition 1 Given a function \( f(x, y) : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) and a point \((x_0, y_0) \in \mathbb{R}^2\), we write

\[
\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L,
\]

if and only if for all \((x, y) \in D\) close enough in distance to \((x_0, y_0)\) the values of \(f(x, y)\) approaches \(L\).
A tool to show that a limit does not exist is the following:

**Theorem 1 (Side limits)** If \( f(x, y) \to L_1 \) along a path \( C_1 \) as \( (x, y) \to (x_0, y_0) \), and \( f(x, y) \to L_2 \) along a path \( C_2 \) as \( (x, y) \to (x_0, y_0) \), with \( L_1 \neq L_2 \), then

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) \text{ does not exist.}
\]

A tool to prove that a limit exists is the following:

**Theorem 2 (Squeeze)** Assume \( f(x, y) \leq g(x, y) \leq h(x, y) \) for all \( (x, y) \) near \( (x_0, y_0) \);

Assume

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L = \lim_{(x, y) \to (x_0, y_0)} h(x, y),
\]

Then

\[
\lim_{(x, y) \to (x_0, y_0)} g(x, y) = L.
\]

Example: How to use the side limit theorem.

- Does the following limit exist?

\[
\lim_{(x, y) \to (0, 0)} \frac{3x^2}{x^2 + 2y^2}.
\]  

(1)

So, the function is \( f(x, y) = \frac{3x^2}{x^2 + 2y^2} \). Let pick the curve \( C_1 \) as the \( x \)-axis, that is, \( y = 0 \). Then,

\[
f(x, 0) = \frac{3x^2}{x^2} = 3,
\]

then

\[
\lim_{(x, 0) \to (0, 0)} f(x, 0) = 3.
\]

Let us now pick up the curve \( C_2 \) as the \( y \)-axis, that is, \( x = 0 \). Then,

\[
f(0, y) = 0,
\]

then

\[
\lim_{(x, 0) \to (0, 0)} f(x, 0) = 0.
\]

Therefore, the limit in (1) does not exist.

Notice that in the above example one could compute the limit for arbitrary lines, that is, \( C_m \) given by \( y = mx \), with \( m \) a constant. Then

\[
f(x, mx) = \frac{3x^2}{x^2 + 2m^2x^2} = \frac{3}{1 + 2m^2},
\]
so one has that
\[ \lim_{(x, mx) \to (0, 0)} f(x, mx) = \frac{3}{1 + 2m^2} \]
is different for each value of \( m \).

Example: How to use the squeeze theorem.

- Does the following limit exist?

\[ \lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^2 + y^2}. \]

Let us first try the side limit theorem, to try to prove that the limit does not exist. Consider the curves \( C_m \) given by \( y = mx \), with \( m \) a constant. Then
\[ f(x, mx) = \frac{x^2 mx}{x^2 + m^2 x^2} = \frac{mx}{1 + m^2}, \]
so one has that
\[ \lim_{(x, mx) \to (0, 0)} f(x, mx) = 0, \quad \forall m \in \mathbb{R}. \]

Therefore, one cannot conclude that the limit does not exist. However, this argument does not prove that the limit actually exists. This can be done with the squeeze theorem.

First notice that
\[ \frac{x^2}{x^2 + y^2} \leq 1, \quad \forall (x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0). \]
(proof: \( 0 \leq y^2 \), then \( x^2 \leq (x^2 + y^2) \)). Therefore, one has the inequality
\[ -|y| \leq \frac{x^2 y}{x^2 + y^2} \leq |y|, \quad \forall (x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0). \]

Then, one knows that \( \lim_{y \to 0} |y| = 0 \), therefore the squeeze theorem says that
\[ \lim_{(x, y) \to (0, 0)} \frac{x^2 y}{x^2 + y^2} = 0. \]
Limits and Continuity

Limit laws for functions of a single variable also holds for functions of two variables.

If the limits \( \lim_{x \to x_0} f \) and \( \lim_{x \to x_0} g \) exist, then

\[
\lim_{x \to x_0} (f \pm g) = \left( \lim_{x \to x_0} f \right) \pm \left( \lim_{x \to x_0} g \right),
\]

\[
\lim_{x \to x_0} (fg) = \left( \lim_{x \to x_0} f \right) \left( \lim_{x \to x_0} g \right).
\]

**Definition 2 (Continuity)** A function \( f(x, y) \) is continuous at \( (x_0, y_0) \) if

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).
\]

Continuity

Examples of continuous functions:

- Polynomial functions are continuous in \( \mathbb{R}^2 \), for example
  \[
P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.
\]

- Rational functions are continuous on their domain,
  \[
f(x, y) = \frac{P_n(x, y)}{Q_m(x, y)},
\]
  for example,
  \[
f(x, y) = \frac{x^2 + 3y - x^2 y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.
\]

- Composition of continuous functions are continuous, example
  \[
f(x, y) = \cos(x^2 + y^2).
\]
Partial derivatives

- Definition of Partial derivatives.
- Higher derivatives.
- Examples of differential equations.

**Definition 3 (Partial derivative)** Consider a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

The partial derivative of $f(x, y)$ with respect to $x$ at $(a, b) \in D$ is denoted as $f_x(a, b)$ and is given by

$$f_x(a, b) = \lim_{h \to 0} \frac{1}{h} [f(a + h, b) - f(a, b)].$$

The partial derivative of $f(x, y)$ with respect to $y$ at $(a, b) \in D$ is denoted as $f_y(a, b)$ and is given by

$$f_y(a, b) = \lim_{h \to 0} \frac{1}{h} [f(a, b + h) - f(a, b)].$$
So, to compute the partial derivative of \( f(x, y) \) with respect to \( x \) at \( (a, b) \), one can do the following: First, evaluate the function at \( y = b \), that is compute \( f(x, b) \); second, compute the usual derivative of single variable functions; evaluate the result at \( x = a \), and the result is \( f_x(a, b) \).

Example:

- Find the partial derivative of \( f(x, y) = x^2 + \frac{y^2}{4} \) with respect to \( x \) at \((1, 3)\).
  1. \( f(1, 3) = 1 + \frac{9}{4}; \)
  2. \( f_x(1, 3) = 2x; \)
  3. \( f_x(1, 3) = 2. \)

To compute the partial derivative of \( f(x, y) \) with respect to \( y \) at \((a, b)\), one follows the same idea. First, evaluate the function at \( x = a \), that is compute \( f(a, y) \); second, compute the usual derivative of single variable functions; evaluate the result at \( y = b \), and the result is \( f_y(a, b) \).

Example:

- Find the partial derivative of \( f(x, y) = x^2 + \frac{y^2}{4} \) with respect to \( y \) at \((1, 3)\).
  1. \( f(1, y) = 1 + \frac{y^2}{4}; \)
  2. \( f_y(1, 3) = y/2; \)
  3. \( f_y(1, 3) = 3/2. \)

Discuss the geometrical meaning of the partial derivative using the graph of the function.

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**Partial derivatives**

The partial derivatives of \( f \) computed at any point \((x, y) \in D\) define the functions \( f_x(x, y) \) and \( f_y(x, y) \), the partial derivatives of \( f \).

More precisely:

**Definition 4** Consider a function \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \subset \mathbb{R} \). The functions partial derivatives of \( f(x, y) \) are denoted by \( f_x(x, y) \) and \( f_y(x, y) \), and are given by the expressions

\[
\begin{align*}
  f_x(x, y) &= \lim_{h \to 0} \frac{1}{h} \left[ f(x + h, y) - f(x, y) \right], \\
  f_y(x, y) &= \lim_{h \to 0} \frac{1}{h} \left[ f(x, y + h) - f(x, y) \right].
\end{align*}
\]
Examples:

- \( f(x, y) = ax^2 + by^2 + xy. \)
  \( f_x(x, y) = 2ax + 0 + y, \)
  \( = 2ax + y. \)
  \( f_y(x, y) = 0 + 2by + x, \)
  \( = 2by + x. \)

- \( f(x, y) = x^2 \ln(y), \)
  \( f_x(x, y) = 2x \ln(y), \)
  \( f_y(x, y) = \frac{x^2}{y}. \)

- \( f(x, y) = x^2 + \frac{y^2}{4}, \)
  \( f_x(x, y) = 2x, \)
  \( f_y(x, y) = \frac{y}{2}. \)

- \( f(x, y) = \frac{2x - y}{x + 2y}, \)
  \( f_x(x, y) = \frac{2(x + 2y) - (2x - y)}{(x + 2y)^2}, \)
  \( = \frac{2x + 4y - 2x + y}{(x + 2y)^2}, \)
  \( = \frac{5y}{(x + 2y)^2}. \)
  \( f_y(x, y) = -\frac{(x + 2y) - (2x - y)^2}{(x + 2y)^2}, \)
  \( = -\frac{5x}{(x + 2y)^2}. \)

- \( f(x, y) = x^3 e^{2y} + 3y, \)
  \( f_x(x, y) = 3x^2 e^{2y}, \)
  \( f_y(x, y) = 2x^3 e^{2y} + 3, \)
  \( f_{yy}(x, y) = 4x^3 e^{2y}, \)
  \( f_{yy}(x, y) = 8x^3 e^{2y}, \)
  \( f_{xy} = 6x^2 e^{2y}, \)
  \( f_{yx} = 6x^2 e^{2y}. \)
Higher derivatives

Higher derivatives of a function $f(x, y)$ are partial derivatives of its partial derivatives. For example, the second partial derivatives of $f(x, y)$ are the following:

\[
\begin{align*}
  f_{xx}(x,y) &= \lim_{h\to0} \frac{1}{h} [f_x(x+h,y) - f_x(x,y)], \\
  f_{yy}(x,y) &= \lim_{h\to0} \frac{1}{h} [f_y(x,y+h) - f_y(x,y)], \\
  f_{xy}(x,y) &= \lim_{h\to0} \frac{1}{h} [f_x(x+h,y) - f_x(x,y)], \\
  f_{yx}(x,y) &= \lim_{h\to0} \frac{1}{h} [f_y(x,y+h) - f_y(x,y)].
\end{align*}
\]
Higher derivatives

**Theorem 3 (Partial derivatives commute)** Consider a function \( f(x, y) \) in a domain \( D \). Assume that \( f_{xy} \) and \( f_{yx} \) exists and are continuous in \( D \). Then,

\[
f_{xy} = f_{yx}.
\]

Examples of differential equations

Differential equations are equations where the unknown is a function, and where derivatives of the function enter into the equation. Examples:

- Laplace equation: Find \( \phi(x, y, z) : D \subset \mathbb{R}^3 \to \mathbb{R} \) solution of
  \[
  \phi_{xx} + \phi_{yy} + \phi_{zz} = 0.
  \]

- Heat equation: Find a function \( T(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R} \) solution of
  \[
  T_t = T_{xx} + T_{yy} + T_{zz}.
  \]

- Wave equation: Find a function \( f(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R} \) solution of
  \[
  f_{tt} = f_{xx} + f_{yy} + f_{zz}.
  \]
Exercises:

- Verify that the function $T(t, x) = e^{-t} \sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t, x) = (t - x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$