Arc length function, Examples

- Review: Arc length of a curve.
- Arc length function.
- Examples Sec. 13.4.

Arc length of a curve

The arc length of a curve in space is a number. It measures the extension of the curve.

Definition 1  The arc length of the curve associated to a vector valued function $\mathbf{r}(t)$, for $t \in [a,b]$ is the number given by

$$\ell_{ba} = \int_a^b |\mathbf{r}'(t)| \, dt.$$  

Suppose that the curve represents the path traveled by a particle in space. Then, the definition above says that the length of the curve is the integral of the speed, $|\mathbf{v}(t)|$. So we say that the length of the curve is the distance traveled by the particle.

The formula above can be obtained as a limit procedure, adding up the lengths of a polygonal line that approximates the original curve.
**Arc length of a curve**

In components, one has,

\[
\mathbf{r}(t) = (x(t), y(t), z(t)), \\
\mathbf{r}'(t) = (x'(t), y'(t), z'(t)), \\
|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}, \\
\ell_{ba} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.
\]

The arc length of a general curve could be very hard to compute.

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**Arc length function**

**Definition 2** Consider a vector valued function \( \mathbf{r}(t) \). The arc length function \( \ell(t) \) from \( t = t_0 \) is given by

\[ \ell(t) = \int_0^t |\mathbf{r}'(u)| du. \]

Note: \( \ell(t) \) is a scalar function. It satisfies \( \ell(t_0) = 0 \).

Note: The function \( \ell(t) \) represents the length up to \( t \) of the curve given by \( \mathbf{r}(t) \).

Our main application: Reparametrization of a given vector valued function \( \mathbf{r}(t) \) using the arc length function.
Arc length function

Reparametrization of a curve using the arc length function:

- With \( \mathbf{r}(t) \) compute \( \ell(t) \), starting at some \( t = t_0 \).
- Invert the function \( \ell(t) \) to find the function \( t(\ell) \).
  Example: \( \ell(t) = 3e^{t/2} \), then \( t(\ell) = 2\ln(\ell/3) \).
- Compute the composition \( \mathbf{r}(\ell) = \mathbf{r}(t(\ell)) \).
  That is, replace \( t \) by \( t(\ell) \).

The function \( \mathbf{r}(\ell) \) is the reparametrization of \( \mathbf{r}(t) \) using the arc length as the new parameter.

Examples, Sec. 13.4

- (Probl. 14 Sec. 13.4) Find the velocity, acceleration, and speed of the position function
  \[
  \mathbf{r}(t) = t \sin(t) \mathbf{i} + t \cos(t) \mathbf{j} + t^2 \mathbf{k}.
  \]
- (Probl. 16 Sec. 13.4) Find the velocity and position vectors given the acceleration and initial velocity and position:
  \[
  \mathbf{a}(t) = -10 \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}.
  \]
- Problems with projectiles. Given the initial speed \( |\mathbf{v}_0| \) and the initial angle of the projectile with the horizontal, \( \theta \), describe the movement of the projectile.
Scalar functions of 2, 3 variables

- Definition.
- Examples.
- Graph of the functions.
- Level curves and level surfaces.

Scalar functions of 2 variables

**Definition 3** A scalar function $f$ of two variables $(x, y)$ is a rule that assigns to each ordered pair $(x, y) \in D \subset \mathbb{R}^2$ a unique real number, denoted by $f(x, y)$, that is,

$$f : D \subset \mathbb{R}^2 \to \mathbb{R}.$$

Comparison:

- Vector valued functions,

  $$r : \mathbb{R} \to \mathbb{R}^2$$
  $$t \to (x(t), y(t))$$

- Scalar function of two variables,

  $$f : \mathbb{R}^2 \to \mathbb{R}$$
  $$(x, y) \to f(x, y).$$
Graph and level curves

Definition 4 The graph of a function $f(x,y)$ is the set of all points $(x,y,z)$ in $\mathbb{R}^3$ of the form $(x,y,f(x,y))$.

Definition 5 The level curves of $f(x,y)$ are the curves in the domain of $f$, $D \subset \mathbb{R}^2$, solutions of the equation

$$f(x,y) = k,$$

for $k \in \mathbb{R}$, a real constant in the range of $f$.

Scalar functions of 3 variables

Definition 6 A scalar function $f$ of three variables $(x, y, z)$ is a rule that assigns to each ordered triple $(x, y, z) \in D \subset \mathbb{R}^3$ a unique real number, denoted by $f(x, y, z)$, that is,

$$f : D \subset \mathbb{R}^3 \rightarrow R \subset \mathbb{R}.$$

Note:

- In order to graph a function $f(x, y, z)$ one needs four space dimensions. So, one cannot do such graph.
- The concept of a level curve can be generalized to functions of more than two variables. In this case they are called level surfaces.
Limits and continuity

- Limit in 2, 3 space dimensions.
- Continuity.
- Examples.

Limits

Idea: The function $f(x, y)$ has the number $L$ as limiting value at the point $(x_0, y_0)$ roughly means that for all points $(x, y)$ near $(x_0, y_0)$ the value of $f(x, y)$ differs little from $L$.

**Definition 7** Consider the function $f(x, y)$ and a point $(x_0, y_0) \in \mathbb{R}^2$. Then,

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if for every number $\epsilon > 0$ there exists another number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ for every $(x, y) \in D$ satisfying

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$
Limits

In words: “$f(x, y)$ has a limit $L$ at $(x_0, y_0)$ if the following holds: for all $(x, y) \in D$ close enough in distance to $(x_0, y_0)$ the values of $f(x, y)$ approaches $L$."

A tool to show that a limit does not exist is the following result.

**Theorem 1** If $f(x, y) \to L_1$ along a path $C_1$ as $(x, y) \to (x_0, y_0)$, and $f(x, y) \to L_2$ along a path $C_2$ as $(x, y) \to (x_0, y_0)$, with $L_1 \neq L_2$, then

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) \text{ does not exist.}$$

**Theorem 2** (Squeeze)

Assume $f(x, y) \leq g(x, y) \leq h(x, y)$ for all $(x, y)$ near $(x_0, y_0)$;

Assume

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L = \lim_{(x, y) \to (x_0, y_0)} h(x, y),$$

Then

$$\lim_{(x, y) \to (x_0, y_0)} g(x, y) = L.$$
Continuity

**Definition 8** A function $f(x, y)$ is continuous at $(x_0, y_0)$ if

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

Examples of continuous functions:

- Polynomial functions are continuous in $\mathbb{R}^2$, for example
  
  $$P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.$$

- Rational functions are continuous on their domain,

  $$f(x, y) = \frac{P_n(x, y)}{Q_m(x, y)},$$

  for example,

  $$f(x, y) = \frac{x^2 + 3y - x^2 y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.$$

- Composition of continuous functions are continuous, example

  $$f(x, y) = \cos(x^2 + y^2).$$