AMCS/MATH 608

Problem set 8 due November 18, 2014

Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Complex Analysis* by Elias Stein and Rami Shakarchi.
Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that f is continuous and there is an A so that $|f(x)| \leq A/(1+x^2)$. Suppose that $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Show that $f \equiv 0$ by using the following argument:

   (a) For each fixed real number t define the functions

   $$A(z) = \int_{-\infty}^{t} f(x)e^{-2\pi i z(x-t)}dx \text{ and } B(z) = -\int_{t}^{\infty} f(x)e^{-2\pi i z(x-t)}dx. \quad (1)$$

   Show that $A(\xi) = B(\xi)$ for all $\xi \in \mathbb{R}$.

   (b) Prove that the function

   $$F(z) = \begin{cases} A(z) \text{ if } \operatorname{Im} z \geq 0, \\ B(z) \text{ if } \operatorname{Im} z < 0 \end{cases} \quad (2)$$

   is entire, and bounded and therefore constant. In fact, show that $F \equiv 0$.

   (c) Deduce that

   $$\int_{-\infty}^{t} f(x)dx = 0, \quad (3)$$

   for all $t \in \mathbb{R}$ and therefore $f \equiv 0$.

2. As usual, let

   $$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx, \quad (4)$$

for $f \in \mathcal{S}(\mathbb{R})$. Show that this linear transformation $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ satisfies the equations $\mathcal{F}^4 = \mathrm{Id}$. For what complex numbers λ can there exist a non-zero function $\varphi \in \mathcal{S}(\mathbb{R})$ so that

$$\mathcal{F}(\varphi) = \lambda\varphi? \tag{5}$$

Show that for some choice of a we have $\mathcal{F}(e^{-a^2x^2}) = e^{-a^2\xi^2}$.

3. Show that for $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{F}\left(\left[\frac{\pm}{\sqrt{2\pi}}\partial_x + \sqrt{2\pi}x\right]\varphi(x)\right) = \pm i\left(\frac{\pm}{\sqrt{2\pi}}\partial_\xi + \sqrt{2\pi}\xi\right)\mathcal{F}(\varphi)(\xi). \tag{6}$$

Use this identity and the result of the previous problem to find eigenfunctions of the Fourier transform for all possible eigenvalues.

4. Suppose that $\{a_0, \dots, a_n\}$ are complex numbers such that $a_n \neq 0$, and the polynomial

$$p(\xi) = \sum_{j=0}^{n} a_j (2\pi i\xi)^j \tag{7}$$

has no real roots.

(a) For $g \in \mathcal{S}(\mathbb{R})$ use the Fourier transform to find a formula for a solution, u_0, to the differential equation

$$\sum_{j=0}^{n} a_j \partial_x^j u(x) = g(x), \tag{8}$$

which tends to zero as $x \to \pm\infty$. Show that, in fact, $u_0 \in \mathcal{S}(\mathbb{R})$.

(b) This equation has an n-dimensional solution space. Describe it as explicitly as you can. Show that the solution, which tends to zero as $\pm x \to \infty$, is unique.

(c) If all the roots of p have positive imaginary parts, and g is compactly supported, then show that there is an A and a $\tau > 0$ so that

$$|u_0(x)| \leq Ae^{-\tau x} \text{ as } x \to \infty. \tag{9}$$

(d) How does the solution, u_0, behave as $x \to -\infty$ if g has compact support and all roots of p lie in the upper half plane?

(e) Assuming that p has no real roots, find the most general condition on $g \in \mathcal{C}_c^\infty(\mathbb{R})$ so that equation (8) has a solution with compact support.

5. Recall that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}. \tag{10}$$

Prove that for $a > 0$, we have

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}. \tag{11}$$

Conclude that the sum equals $\coth(\pi a)$. Prove that for any $z \in \mathbb{C} \setminus \{in : n \in \mathbb{Z}\}$ we have the identity

$$\coth(\pi z) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{z}{z^2+n^2}. \tag{12}$$

deduce from this formula that

$$\tanh\left(\frac{\pi z}{2}\right) = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{z}{(2n-1)^2+z^2}. \tag{13}$$