AMCS/MATH 608
Problem set 7 due November 11, 2014
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Complex Analysis* by Elias Stein and Rami Shakarchi.
Standard problem This does not have to be turned in:

1. Show that there exists a surjective holomorphic map from the unit disk onto $\mathbb{C}$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let f be a holomorphic function in a sector, S, whose vertex is the origin with an openning angle π/β, which extends continuously to $\overline{S}$. Suppose that $|f(z)| \leq 1$ on bS and, for some constants c, C, and an $0 < \alpha < \beta$, we also have the estimate.
$$|f(z)| \leq Ce^{c|z|^{\alpha}}. \tag{1}$$
Show that
$$|f(z)| \leq 1 \text{ for } z \in S. \tag{2}$$

2. Let $D = \mathbb{C} \setminus (-\infty, 0]$. Show that if $f_{\pm}(x)$ are bounded continuous functions on $(-\infty, 0]$, with $f_+(0) = f_-(0)$, then there is a unique bounded harmonic function u defined in D such that
$$\lim_{y\to 0^+} u(x, y) = f_+(x) \text{ and } \lim_{y\to 0^-} u(x, y) = f_-(x) \text{ for all } x \in (-\infty, 0]. \tag{3}$$

3. Show that if $f : D_R(0) \to \mathbb{C}$ is holomorphic, and $|f(z)| < M$, for some positive M, then
$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leq \frac{|z|}{MR}, \text{ for all } z \in D_R(0). \tag{4}$$

4. The pseudo-hyperbolic distance between two points in $D_1(0)$ is defined to be
$$\rho(z, w) = \left|\frac{z - w}{1 - \overline{w}z}\right|. \tag{5}$$

(a) Prove that if $f : D_1(0) \to D_1(0)$ is holomorphic, then

$$\rho(f(z), f(w)) \leq \rho(z, w) \text{ for all } z, w \in D_1(0). \tag{6}$$

Show also that if f is 1-1 and onto, then it preserves this distance, i.e., for such maps $\rho(f(z), f(w)) = \rho(z, w)$ for all $z, w \in D_1(0)$.

(b) For general holomorphic maps $f : D_1(0) \to D_1(0)$, prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{7}$$

5. Suppose that $\xi_1 < \xi_2 < \cdots < \xi_n$ are points on the real axis and that

$$f(z) = \int_0^z \frac{dw}{[(w - \xi_1)(w - \xi_2) \cdots (w - \xi_n)]^{\frac{2}{n}}}, \tag{8}$$

maps the upper half plane 1-1, onto a regular convex polygon. Give a precise description for all possible choices of the points $\{\xi_1, \ldots, \xi_n\}$. Hint: Start by showing that

$$f(z) = \int_0^z \frac{dw}{(1 - w^n)^{\frac{2}{n}}}, \tag{9}$$

maps the unit disk to a regular n-gon.

6. Let $0 < b \leq a$ be positive numbers, and define the pair (a_1, b_1) to be the arithmetic and geometric means of a and b

$$a_1 = \frac{a + b}{2} \text{ and } b_1 = \sqrt{ab}. \tag{10}$$

Repeating these operations defines a sequence of pairs $< (a_n, b_n) >$.

(a) Prove that the two sequences $< a_n >$ and $< b_n >$ have a common limits, $M(a, b)$, which we call the arithmetic-geometric metric of (a, b). Hint: show that

$$a \geq a_1 \geq a_2 \geq \cdots a_n \geq b_n \cdots \geq b_1 \geq b \text{ and } a_n - b_n \leq \frac{a - b}{2^n}. \tag{11}$$

In fact, the sequence $a_n - b_n$ tends to zero much faster than indicated by this estimate. Can you find a more accurate result, assuming that $b < a < 2b$? You should write a program to compute these iterates and the differences $a_n - b_n$. Use the $\log(a_n - b_n)$, so you can see how fast it is going to zero.

(b) Gauss proved that

$$\frac{1}{M(a,b)} = \frac{2}{\pi} \int\limits_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}}. \tag{12}$$

Prove this relation by:

i. Showing that it follows from the fact that the integral on the right hand side of (12), $I(a,b)$, satisfies

$$I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right). \tag{13}$$

ii. Next show that, with $k^2 = 1 - \frac{b^2}{a^2}$, we have

$$I(a,b) = \frac{1}{a} K(k) = \frac{1}{a} \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{14}$$

iii. Finally show that (13) is equivalent to the identity

$$K(k) = \frac{2}{1+k'} K\left(\frac{1-k'}{1+k'}\right) \text{ where } k'^2 = 1 - k^2. \tag{15}$$

iv. Give a geometric interpretation for the number $K(k)$.

7. For $1 < \rho$, define an analytic function on the upper half plane by:

$$f(z) = \int_0^z \frac{dw}{\sqrt{w(1-w)(\rho - w)}}. \tag{16}$$

Show that, for any choice of $1 < \rho$, the image of the upper half plane is a rectangle. What are the side lengths of this rectangle?