AMCS/MATH 608
Problem set 6 due October 28, 2014
Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari. Conformal Mapping is an especially good reference for material on harmonic functions.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. A polynomial $p(x, y)$ is homogeneous of degree k if $\lambda \in (0, \infty)$,
$$p(\lambda x, \lambda y) = \lambda^k p(x, y) \text{ for all } (x, y) \in \mathbb{R}^2. \tag{1}$$
Any polynomial p of degree can be written as sum of homogeneous polynomials. Prove that a polynomial in (x, y) is harmonic if and only if each homogeneous part is harmonic. For each $n \in \mathbb{N}$ find a basis for the two dimensional, real vector space, $\mathcal{H}_n$, of homogeneous, harmonic polynomials of degree n. You must prove that $\dim \mathcal{H}_n = 2$. Suppose that p is a homogeneous polynomial of degree n, show that there are harmonic polynomials $\{h_j\}$ of degrees $j \in \{n, n-2, \dots, 2, 0\}$, if n is even, and $j \in \{n, n-2, \dots, 3, 1\}$, if n is odd, so that
$$p(x, y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} r^{2j} h_{n-2j}(x, y), \tag{2}$$
where $r^2 = x^2 + y^2$.

2. Suppose that D is a simply connected domain in $\mathbb{C}$ for which there is an 1-1, onto analytic map $f : D \to D_1(0)$, which extends to be a 1-1, onto, C^1-map from $\overline{D}$ $\overline{D}_1(0)$. Let $g \in C^0(bD)$; prove that
$$u(z, \bar{z}) = \frac{1}{2\pi} \int\limits_{bD} g(w) \frac{1 - |f(z)|^2}{|f(w) - f(z)|^2} \cdot \frac{f'(w)dw}{if(w)}, \tag{3}$$

solves Dirichlet's problem in D,

$$\begin{cases} \Delta u = 0 \text{ in } D, \\ u \restriction_{bD} = g. \end{cases} \tag{4}$$

Give a geometric interpretation for the measure

$$ds = \frac{f'(w)dw}{if(w)} \text{ for } w \in bD, \tag{5}$$

which explains why it is real. If γ is a connected arc of bD, then describe, in simple geometric terms, the value of the integral:

$$\frac{1}{2\pi} \int_{\gamma} \frac{f'(w)dw}{if(w)}. \tag{6}$$

3. Show that if u is harmonic on $D_R(0)$ and continuous on $\overline{D_R(0)}$, then

$$u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - |z|^2)u(Re^{i\theta}, Re^{-i\theta})}{|Re^{i\theta} - z|^2} d\theta. \tag{7}$$

Prove (7) and show that this implies:

$$\partial_z u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-i\theta} u(Re^{i\theta}, Re^{-i\theta})}{R} d\theta. \tag{8}$$

Show that a bounded harmonic function defined in the whole complex plane is constant. Let u be a bounded harmonic function defined in a domain D. Show that the gradient of u satisfies the estimate

$$|\nabla u(x, y)| \leq \frac{2M}{\text{dist}((x, y), D^c)}, \tag{9}$$

provided $|u(z)| \leq M$ in D.

4. Using the formula in (7) with $R = 1$, prove the following statement: If u is a non-negative harmonic function in $D_1(0)$, which is continuous up to $bD_1(0)$, then

$$\max_{z \in B_{\frac{1}{2}}(0)} u(z, \bar{z}) \leq 9 \min_{z \in B_{\frac{1}{2}}(0)} u(z, \bar{z}). \tag{10}$$

Use this estimate to show, without using complex function theory, that a positive harmonic function defined in the whole complex plane is constant. Hint: Consider $u(Rz, R\bar{z})$.

5. Let $< u_n >$ be a sequence of harmonic functions defined in a connected open set D. Suppose that $< u_n >$ converges locally uniformly to a function u. Prove that the limit is also harmonic. Do **not** use the harmonic conjugate. Hint: This is a local property.

6. Let u be a continuous function defined in a connected open set D. Let $z \in D$, and suppose that $r_z = \text{dist}(z, D^c)$. We say that u satisfies the mean value property in D if, for every $z \in D$ and $r < r_z$ we have that

$$u(z, \bar{z}) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}, \bar{z} + re^{-i\theta}) d\theta. \tag{11}$$

   (a) Prove that a C^2-function that satisfies the mean value property is harmonic.

   (b) Prove that a C^0-function u defined in D satisfying the mean value property is actually harmonic, that is, has 2 continuous derivatives and satisfies $\Delta u = 0$. Hint: If $\varphi(x, y)$ is a smooth function with "small support," then, where it is defined, the function

$$u * \varphi(x, y) = \int_{\mathbb{R}^2} u(x - x', y - y')\varphi(x', y')dx'dy' \tag{12}$$

   is smooth. Take φ to be a radial function.