AMCS/MATH 608
Problem set 5 due October 21, 2014
Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that D is a bounded connected region in the plane, with a C^1 boundary, and u, v are twice continuously differentiable functions in D, whose first derivatives have continuous extensions to $\overline{D}$. We let $\boldsymbol{n}$ denote the outer unit normal vector along bD, and $\boldsymbol{t} = i\boldsymbol{n}$, the oriented unit tangent vector. The normal and tangential derivatives of u along the boundary are defined by:

$$\frac{\partial u}{\partial \boldsymbol{n}} = \langle \nabla u, \boldsymbol{n} \rangle \quad \frac{\partial u}{\partial s} = \langle \nabla u, \boldsymbol{t} \rangle. \tag{1}$$

   (a) Show that *Stokes' Theorem* (for 1-forms) implies that

$$\int\limits_D [u_x v_x + u_y v_y] dxdy + \int\limits_D u\Delta v dxdy = \int\limits_{bD} u \frac{\partial v}{\partial \boldsymbol{n}} ds. \tag{2}$$

   Here ds denotes arclength measure along bD.

   (b) Use this formula to deduce that if u is also harmonic in D, then

$$\int\limits_{bD} \frac{\partial u}{\partial \boldsymbol{n}} ds = 0. \tag{3}$$

   (c) Show that equation (2) implies that

$$\int\limits_D [u\Delta v - v\Delta u] dxdy = \int\limits_{bD} [u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}] ds. \tag{4}$$

2. Suppose that h is a continuous function on $\mathbb{R}$ with support in the finite interval $[-1, 1]$, and define

$$g(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{h(x)dx}{x - z} \tag{5}$$

   (a) Prove that g is an analytic function in $\mathbb{C} \setminus [-1, 1]$, which tends to zero as $|z|$ tends to infinity.

   (b) Let $D_1^+(0)$ ($D_1^-(0)$) denote the upper (resp. lower) half of the unit disk. If h is the restriction of an analytic function defined in $D_1(0)$, then show that g has analytic continuations, across the interval $(-1, 1)$, from $D_1^+(0)$ to $D_1(0)$ and from $D_1^-(0)$ to $D_1(0)$. Hint: Deform the contour of integration $[-1, 1]$ to a curve lying below (resp. above) $(-1, 1)$.

   (c) Does the continuation of g from $D_1^+(0)$ to $D_1^-(0)$, defined in (b), ever agree with $g \restriction_{D_1^-(0)}$, as defined by (5)?

3. In class we proved that for any $\varphi \in \mathscr{C}_c^1(\mathbb{R}^2)$ the equation

$$\partial_{\bar{z}} u = \varphi \tag{6}$$

   has a solution given by

$$u_0(z, \bar{z}) = \frac{1}{\pi} \iint \frac{\varphi(w, \bar{w})dxdy}{w - z}. \tag{7}$$

   (a) u_0 is one solution; what are all the other solutions?

   (b) Show that this solution satisfies $\lim_{z\to\infty} u_0(z, \bar{z}) = 0$, and it is uniquely determined by this condition.

   (c) What conditions must φ satisfy for there to exist a solution with compact support? Hint: The solution u_0 is analytic outside the support of φ. Find a representation, for z with large modulus, that reflects this fact. Note that φ must satisfy infinitely many conditions.

4. Using (4) show that if u is a C^2-function with compact support then

$$\frac{1}{4\pi} \int_{\mathbb{C}} \log(x^2 + y^2) \Delta u(x, y) dxdy = u(0). \tag{8}$$

If u is a compactly supported, C^2-function, then show that the function

$$U(x,y) = \frac{1}{4\pi}\int\limits_{\mathbb{C}} \log(x'^2 + y'^2)u(x-x', y-y')dx'dy' \tag{9}$$

is a twice differentiable function satisfying

$$\Delta U = u. \tag{10}$$

Hint: Be careful because $\log(x^2+y^2)$ is singular at $(0,0)$.

Under what condition is

$$\lim_{(x,y)\to\infty} U(x,y) = 0? \tag{11}$$

5. Suppose that u is a harmonic function defined in a simply connected domain, D with C^1-boundary, and let v denote a harmonic conjugate to u. Suppose that the first derivatives of u and v extend continuously to the bD. For a differentiable function f defined along bD let $\frac{\partial f}{\partial s}$ denote the derivative of f with respect to arclength along bD. If $\boldsymbol{t}$ is the unit tangent vector to bD, oriented in the positive direction, then the tangential derivative is:

$$\frac{\partial(v \restriction_{bD})}{\partial s} = \langle \nabla v, \boldsymbol{t}\rangle \restriction_{bD} . \tag{12}$$

   (a) Show that along bD we have the relation:

$$\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial v}{\partial s}. \tag{13}$$

   (b) Let $g(s)$ be a continuous function defined along $bD_1(0)$ with s the arclength parameter and

$$\int\limits_{bD_1(0)} g(s)ds = 0. \tag{14}$$

   Explain how to use (13) to prove that there is a harmonic function u defined in $D_1(0)$ such that

$$\frac{\partial u}{\partial \boldsymbol{n}}(s) = g(s). \tag{15}$$

   Hint: The function $G(s) = \int\limits_{s_0}^{s} g(\sigma)d\sigma$ is a continous function on bD_1.

6. Let $g(x, y) = \frac{1}{4\pi}\log(x^2 + y^2)$. Let $\gamma : [0, L] \to \mathbb{C}$ give an arclength parametrization of, Γ, a simple closed $\mathscr{C}^1$-curve. Let D be the domain bounded by Γ. For σ, a continuous function defined on Γ, and ds the arclength measure along Γ we define the function

$$u(x, y) = \int_0^L g(x - \gamma_1(s), y - \gamma_2(s))\sigma(s)ds, \quad (x, y) \in \mathbb{C}. \tag{16}$$

(a) Show that u is a continuous function in $\mathbb{C}$, which is harmonic in $\mathbb{C} \setminus \Gamma$.

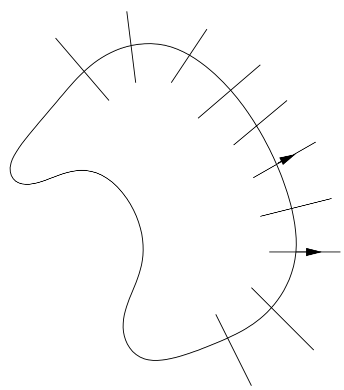

Figure 1. Figure showing contour Γ with several normal lines, and outward normal vectors.

(b) Let $\boldsymbol{n}$ denote the outward unit normal vector field along Γ. The normal lines to Γ foliate a neighborhood of Γ, see Figure 1. We can therefore extend $\boldsymbol{n}$ to a neighborhood of Γ as the unit tangent directions to these lines. For $p \in \Gamma$, let $\partial_{\boldsymbol{n}}^{+}u(p)$ denote the limit of $\partial_{\boldsymbol{n}}u(q)$ from the inside of D, as $q \to p$, and $\partial_{\boldsymbol{n}}^{-}u(p)$ the analogous limit from D^c. What is

$$\partial_{\boldsymbol{n}}^{+}u(p) - \partial_{\boldsymbol{n}}^{-}u(p)? \tag{17}$$