AMCS/MATH 608
Problem set 4 due October 7, 2014
Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari. Conformal Mapping is an especially good reference for material on harmonic functions.

Standard problems: The following problems should be done, but do not have to be handed in.

1. For each $z \in \mathbb{C}$ the function

$$g(z;t) = \exp\left(\frac{z}{2}(t - t^{-1})\right) \tag{1}$$

is an analytic function of $t \in \mathbb{C} \setminus \{0\}$ and therefore has a Laurent expansion:

$$g(z;t) = \sum_{n=-\infty}^{\infty} J_n(z)t^n, \tag{2}$$

convergent in this set.

(a) Show that

$$J_n(z) = \frac{1}{2\pi i} \int\limits_{\{t\colon |t|=1\}} \frac{g(z;t)dt}{t^{n+1}}. \tag{3}$$

Conclude that $J_n(z)$ is an entire function of z for every $n \in \mathbb{Z}$, and that $J_{-n}(z) = J_n(-z)$.

(b) Use this contour integral to estimate $J_n(z)$. Can you show that there is a constant C so that

$$|J_n(z)| \leq C\sqrt{|n|+1}\frac{\left(\frac{|z|}{2}\right)^{|n|} e^{|z|}}{|n|!}? \tag{4}$$

Hint: You will need to choose a contour in (3) that depends on z to get the optimal result. Why is this allowed?

(c) Use the generating function representation to show that:

$$J_n''(z) + \frac{1}{z} J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0. \tag{5}$$

2. Evaluate the integral $\int\limits_0^\infty \frac{dx}{1+x^4}$.

3. Let $\{z_1, \ldots, z_n\}$ be points lying inside the simple closed curve γ and define

$$p(z) = \prod_{j=1}^{n} (z - z_j). \tag{6}$$

If f is analytic inside of γ and continuous up to γ, then show that

$$P(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f(\zeta)}{p(\zeta)} \frac{p(\zeta) - p(z)}{\zeta - z} d\zeta \tag{7}$$

is a polynomial of degree $n - 1$, which satisfies:

$$P(z_j) = f(z_j) \text{ for } j = 1, \ldots, n. \tag{8}$$

4. Prove that the sequence of entire functions, $f_n(z) = (1 + \frac{z}{n})^n$ converges locally uniformly to $f(z) = e^z$. Using *this fact*, prove that $f(z) = 0$ has no solution.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Show that, if a is a real number greater than 1, then the equation

$$ze^{a-z} = 1 \tag{9}$$

has precisely one root in the unit disk $\{z : |z| \leq 1\}$. Explain why this root is necessarily a positive real number.

2. The Beta function is defined for $\text{Re}(\alpha) > 0$ and $\text{Re}(\beta) > 0$ by

$$B(\alpha, \beta) = \int\limits_0^1 (1 - t)^{\alpha - 1} t^{\beta - 1} dt. \tag{10}$$

Prove that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{11}$$

Hint: Note that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds, \tag{12}$$

and let $s = ur$, and $t = u(1-r)$.

3. Evaluate the following integrals:

   (a) $\int_0^\infty \frac{x^{\alpha-1}dx}{(x+\beta)(x+\gamma)}$, where $0 < \alpha < 1$, and $\beta, \gamma > 0$.

   (b) $\int_0^\infty \frac{\cos x dx}{a^2+x^2}$, for $a > 0$.

   (c) $\int_{-\infty}^\infty \frac{x \sin x dx}{a^2+x^2}$, for $a > 0$. Note that this integral is not absolutely convergent, so you need to specify the meaning of this integral as a limit of integrals over finite intervals.

   (d) Show that, for $\in \mathbb{N}$, we have

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.$$

4. Compute the improper Riemann integrals:

$$\int_0^\infty \cos(x^2) dx = \lim_{r\to\infty} \int_0^r \cos(x^2) dx, \quad \int_0^\infty \sin(x^2) dx = \lim_{r\to\infty} \int_0^r \sin(x^2) dx, \tag{13}$$

by evaluating the contour integral,

$$\lim_{r\to\infty} \int_{\Gamma_r} e^{-z^2} dz, \tag{14}$$

where Γ_r is the contour shown in Figure 1.

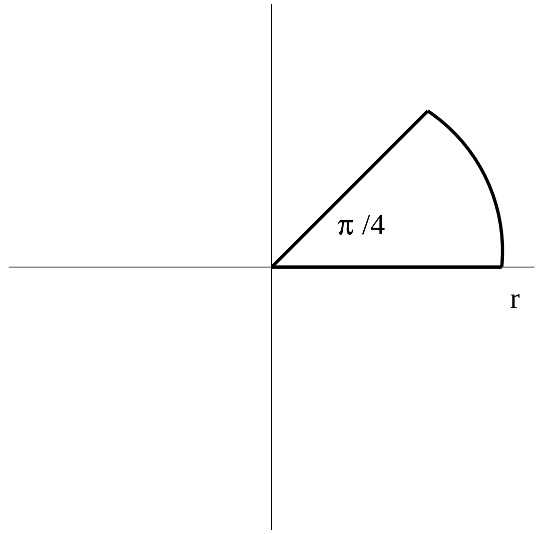

Figure 1. The integration contour Γ_r.

5. Let f be a 1-1 analytic function defined in $D_1(0)$, with $f(0) = w_0$. As shown in class, the inverse of f for w a neighborhood of w_0 is given by

$$g(w) = \frac{1}{2\pi i} \int_{|z|=1} \frac{z f'(z)}{f(z) - w} dz. \tag{15}$$

Use the fact that

$$\frac{1}{f(z) - w} = \frac{1}{f(z) - w_0 - (w_0 - w)}, \tag{16}$$

to derive a formula, in terms of f, for the Taylor coefficients of g at w_0.

6. Show that if $|a| < 1$, then

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta = 0. \tag{17}$$

7. Suppose that f is an non-vanishing analytic function defined in a connected domain $D \subset \mathbb{C}$. Suppose that for every $n \in \mathbb{N}$, there is an analytic function $h_n(z)$ defined in D so that $f(z) = [h_n(z)]^n$. Show that there is an analytic function $g(z)$ defined in D so that $f(z) = e^{g(z)}$. Note: we do NOT assume that D is simply connected.