AMCS/MATH 608
Problem set 3 due September 30, 2014
Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. A region $D \subset \mathbb{C}$ is simply connected if every closed curve in D can be continuously deformed to a point through a family of cloed curves contained in D. For $0 \leq r < R$ show that the annular region:

$$A_{rR} = \{z : r < |z| < R\} \tag{1}$$

is not simply connected.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let f be a funtion defined and analytic in the right half plane $H = \{\text{Re}\, t \geq 0\}$ and suppose that there is a constant C so that

$$|f(t)| \leq \frac{C}{1 + |t|^2} \text{ for } t \in H. \tag{2}$$

Define the function:

$$F(x) = \int\limits_{\{\text{Re}\, t=0\}} f(t)e^{tx}dt. \tag{3}$$

Suppose that for some $0 < \theta < \frac{\pi}{2}$, and $0 < R$, f extends to be analytic in the set

$$H \cup \{t : |\arg t| \leq \theta + \frac{\pi}{2} \text{ and } |t| > R\}, \tag{4}$$

where it continues to satisfy the estimate in (2). Show that F extends to define an analytic function $F(z)$, in the set

$$\{z : |\arg z| < \theta\}. \tag{5}$$

Hint: Consider the contours $\Gamma_{R,\phi}$ shown below.

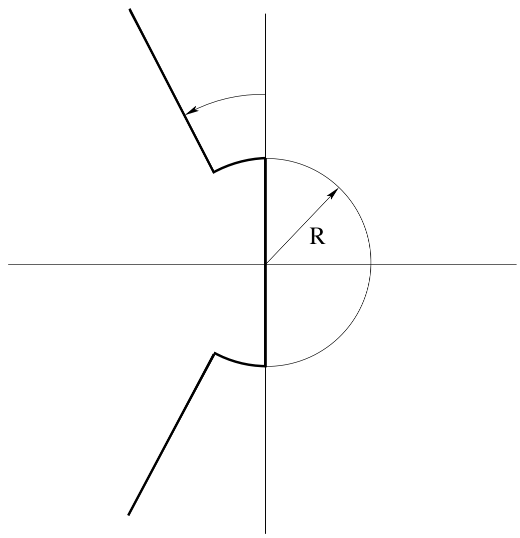

Figure 1. The contour $\Gamma_{R,\phi}$.

2. Suppose that $\alpha \in D_1(0)$, and $\theta \in \mathbb{R}$. Show that
$$f(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}, \tag{6}$$
is a 1-1, onto analytic map of $D_1(0)$ to itself. Show that every 1-1, onto analytic self map of $D_1(0)$ is of this form. Show that $g(z) = i(1+z)/(1-z)$ is a 1-1, onto analytic map from the unit disk to $H_+ = \{z : \operatorname{Im} z > 0\}$. Use this map and the first part of the problem to find all the 1-1, onto analytic maps of H_+ to itself. Be as explicit as you can be.

3. Suppose that f is a non-vanishing analytic function in $D_1^+(0)$ that extends continuously to the set $D_1^+(0) \cup (-1, 1)$. Suppose that for $x \in (-1, 1)$, the value $f(x)$ lies in $bD_R(0)$; show that defining $f(z)$, for $z \in D_1^-(0)$, by
$$f(z) = \frac{R^2}{\overline{f(\bar{z})}}, \tag{7}$$
gives an analytic continuation of f to $D_1(0)$. You can assume that f does not vanish in $D_1^+(0)$.

4. Prove that if f is an analytic function in all of $\mathbb{C}$, except for poles, and f has, at worst, a pole at infinity, then there are polynomials p and q so that
$$f(z) = \frac{p(z)}{q(z)}. \tag{8}$$

Note: We say that "$f(z)$ has, at worst, a pole at ∞" if $f(1/z)$ has, at worst, a pole at $z = 0$.

5. Let $U \subset \mathbb{C}$ be an open set. For f a function defined in U we define the norms:

$$\|f\|_{L^2(U)} = \sqrt{\iint\limits_U |f(z,\bar{z})|^2 dxdy}, \tag{9}$$

and

$$\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z,\bar{z})|. \tag{10}$$

Suppose that f is holomorphic in $D_1(0)$ show that, for each $0 < s < r < 1$, there is a constant C_{rs} (depending on r, s, but not on f) so that

$$\|f\|_{L^\infty(D_s(0))} \leq C_{rs}\|f\|_{L^2(D_r(0))}. \tag{11}$$

Suppose that $< f_n >$ is a sequence of analytic functions, with finite $L^2(B_1(0))$-norms, for which there is a function $f \in L^2(D_1(0))$, such that

$$\lim_{n \to \infty} \|f - f_n\|_{L^2(D_1(0))} = 0. \tag{12}$$

Prove that the limit function f is also analytic in $D_1(0)$, or more precisely, has a representative that is analytic in $D_1(0)$. Show that $\|f\|_{L^2(D_1(0))} < \infty$.

6. Suppose that f is an analytic function in $D_{1+\delta}(0) \setminus \{z_0\}$, where $\delta > 0$ and $|z_0| = 1$. Show that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{13}$$

in the unit disk, and f has at worst a pole at z_0, then

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0. \tag{14}$$