AMCS/MATH 608

Problem set 2 due September 23, 2014

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Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Show that there does not exist an analytic function in $D_1(0)$, which extends continuously to $\{z : |z| = 1\}$, so that $f(z) = 1/z$ on the unit circle.

2. Suppose that $f : D_1(0) \to \mathbb{C}$ is an analytic map and $f'(0) \neq 0$. In class we showed that there is an $r > 0$ and an *analytic* map $g : D_r(f(0)) \to D_1(0)$, satisfying $g(f(0)) = 0$, $f(g(w)) = w$, for $w \in D_r(f(0))$, and $g(f(z)) = z$, for z closed enough to 0.

   (a) Show that for any $n \in \mathbb{N}$ and $w_0 \in \mathbb{C} \setminus \{0\}$ there is an analytic nth root function $g_n(w)$ defined in a neighborhood U of w_0, so that, for $w \in U$, we have $(g_n(w))^n = w$. Explain why, if $n > 1$, no such function can be analytic in a neighborhood of 0.

   (b) Show that there is a neighborhood U of any point $w_0 \in \mathbb{C} \setminus \{0\}$ in which an analytic function $l(w)$ is defined that satisfies

   $$e^{l(w)} = w, \tag{1}$$

   for $w \in U$. This is a branch of the log. What is the real part of l? For this problem we can take e^z to be the entire function defined by the power series:

   $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2}$$

   You need to prove that e^z satisfies the hypotheses in part (a).

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $\{w_1, \ldots, w_m\}$ be points in the unit circle. Show that there exists a point, z on the unit circle where
$$\prod_{j=1}^{m} |z - w_j| > 1. \tag{3}$$
Conclude that there are also points on the unit circle where
$$\prod_{j=1}^{m} |z - w_j| = 1. \tag{4}$$

2. Show that
$$4\partial_z \partial_{\bar{z}} = 4\partial_{\bar{z}} \partial_z = \Delta, \tag{5}$$
where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator.

   (a) Show that if $f = u + iv$ is an analytic function in an open subset of $\mathbb{C}$, then u and v are harmonic, that is
$$\Delta u = \Delta v = 0. \tag{6}$$

   (b) Suppose that U is a harmonic function in $B_1(0)$. Show that there is a function V that satisfies the system of equations:
$$V_x = -U_y \text{ and } V_y = U_x. \tag{7}$$
Conclude that $U + iV$ is analytic in $B_1(0)$. If $U = x^5 - 10x^3y^2 + 5xy^4$ what is V?

3. Suppose that f is analytic in $D_{R_0}(0)$. Show that whenever $0 < R < R_0$ and $|z| < R$, then
$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \operatorname{Re}\left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right) d\theta. \tag{8}$$
Show that
$$\operatorname{Re}\left(\frac{Re^{i\theta} + r}{Re^{i\theta} - r}\right) = \frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2}. \tag{9}$$

Finally, if $f = u + iv$, and $f(0)$ is real, then show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) d\theta. \tag{10}$$

4. Suppose that f is analytic in the set $D_1(0)^c = \{z : |z| > 1\}$, continuous up to $|z| = 1$, and bounded as $z \to \infty$. Find an analogue of the Cauchy integral formula expressing $f(z)$ for $z \in D_1(0)^c$ as a complex weighted average of the values of f on the unit circle.

   Using this formula find a simple expression for $\lim_{z\to\infty} f(z)$. How do we know that this limit exists?

   Find a similar formula for $f(z)$, $z \in D_1(0)^c$, if f satisfies an estimate of the form $|f(z)| \leq C|z|^n$ as $z \to \infty$.

   You must prove that your formulæ are correct.

5. Let f be a non-constant analytic function defined in a neighborhood of the closed unit disk. Suppose that $|f(z)| = 1$ where $|z| = 1$. Show that $\theta \to f(e^{i\theta})$ goes counterclockwise around the unit disk, and makes at least one full rotation. Hint: In a neighborhood of any point on the unit disk $\log f(z) = \log|f(z)| + i \arg f(z)$ is an analytic function. Use the maximum principle.

6. Suppose that f is an analytic function defined in $D_1(0)$. We say that $e^{i\theta}$ is a regular boundary point for f if there is $\rho > 0$, so that f has an analytic extension to the set $D_1(0) \cup D_\rho(e^{i\theta})$. The function

$$f(z) = \sum_{j=0}^{\infty} z^{2^j}, \tag{11}$$

   is clearly analytic in $D_1(0)$. Prove that there are no regular boundary points. Hint: consider points of the form $re^{i\theta}$ where $\theta = \frac{2\pi p}{2^k}$, with p and k positive integers.