AMCS/MATH 608
Problem set 11 due December 16, 2014
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi.

Standard problems: The solutions to the following problems do not need to be handed in.

1. Suppose that f is a non-negative integable function, and define $E_\alpha = \{x : f(x) \geq \alpha\}$. Prove that
$$m(E_\alpha) \leq \frac{1}{\alpha} \int f(x) dm(x) \tag{1}$$
This is Chebyshev's Inequality, which is quite important in Probability Theory.

2. Given a collection of sets $\{F_1, \ldots, F_m\}$ show that there is a collection of disjoint sets $\{F_1^*, \ldots, F_N^*\}$, so that $N \leq 2^m - 1$, and for every k we have that
$$F_k = \bigcup_{F_j^* \subset F_k} F_j^*. \tag{2}$$

3. Suppose that f is an integrable function on $\mathbb{R}^d$, and that for every measurable set E we know that
$$\int_E f(x) dx \geq 0. \tag{3}$$
Prove that $f(x) \geq 0$ for almost every x.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. The integrability of a function on $\mathbb{R}$ does not imply that it tends to zero at infinity.
   (a) Construct a positive continuous function on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but $\limsup_{x \to \infty} f(x) = \infty$.
   (b) If f is integrable and uniformly continous on $\mathbb{R}$, then show that $\lim_{x \to \pm\infty} f(x) = 0$.

2. Suppose that f is integrable on $\mathbb{R}$. Show that

$$g(x) = \int_{-\infty}^{x} f(y)dy \tag{4}$$

is uniformly continuous.

3. Suppose that f is a non-negative measurable function, and for each $k \in \mathbb{Z}$ define the sets:

$$E_k = \{x : f(x) > 2^k\} \text{ and } F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}. \tag{5}$$

Show that the sets $\{F_k\}$ are pairwise disjoint and that if f is finite a.e., then

$$\{x : f(x) > 0\} = \bigcup_{k=-\infty}^{\infty} F_k. \tag{6}$$

Prove that f is is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \text{ if and only if } \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty. \tag{7}$$

For $a \in \mathbb{R}$ define the functions on $\mathbb{R}^d$ by

$$f_a(x) = \begin{cases} |x|^{-a} \text{ if } |x| \leq 1, \\ 0 \text{ if } |x| > 1. \end{cases} \tag{8}$$

$$g_a(x) = \begin{cases} |x|^{-a} \text{ if } |x| \geq 1, \\ 0 \text{ if } |x| < 1. \end{cases} \tag{9}$$

Using the second part of the problem show that f_a is integrable in $\mathbb{R}^d$ if and only if $a < d$, while g_a is integrable in $\mathbb{R}^d$ if and only if $a > d$.

4. Define a function on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} x^{-\frac{1}{2}} \text{ if } 0 < x < 1, \\ 0 \text{ otherwise.} \end{cases} \tag{10}$$

Let $\{r_n\}$ be an enumeration of the rational numbers, $\mathbb{Q}$, and define

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n). \tag{11}$$

Prove that F is integrable over $\mathbb{R}$, hence the series defining $F(x)$ converges for a.e. x. However, show that this series is unbounded on every interval, and that any function that agrees with F a.e. is as well.

5. Let f be a bounded, non-decreasing function defined on $[0, 1]$, that is, if $x < y$, then $f(x) \leq f(y)$. Show that f is Riemann integrable.

6. Suppose that $0 \leq f$ is integrable over $\mathbb{R}^d$. For each $\alpha > 0$ define

$$E_\alpha = \{x : f(x) > \alpha\}. \tag{12}$$

Prove that

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty m(E_\alpha)d\alpha. \tag{13}$$

Hint: Think of the right hand side as a Riemann integral, in particular, show that $\alpha \mapsto m(E_\alpha)$ is Riemann integrable.