AMCS/MATH 608
Problem set 10 due December 9, 2014
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi.

Standard problems: The solutions to the following problems do not need to be handed in.

1. If f is a function then we define
$$f^+(x) = \max\{0, f(x)\} \text{ and } f^-(x) = \min\{0, f(x)\}. \tag{1}$$
Show that if f is measurable, then so are f^+ and f^-, and therefore so is $|f|$.

2. Show that f is measurable if and only if the sets $\{x : f(x) \geq a\}$ are measurable for every $a \in \mathbb{R}$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that a σ-algebra is either finite or uncountable. Give an example of a finite σ-algebra.

2. Prove that every measurable function is the limit a.e. of a sequence of continuous functions.

3. Let $D \subset \mathbb{R}$ be a dense subset. Let f be an extended real-valued function defined on $\mathbb{R}$. Show that if the sets $\{x : f(x) > a\}$ are measurable for all $a \in D$, then f is measurable.

4. Let $E \subset \mathbb{R}^d$ be a measurable set and f a function defined on E. We define the function
$$g(x) = \begin{cases} f(x) \text{ if } x \in E, \\ 0 \text{ if } x \notin E. \end{cases} \tag{2}$$
Show that f is measurable if and only if g is measurable.

5. Let $\{f_n\}$ be a sequence of measurable functions on $[0, 1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there is a sequence of $\{c_n\}$ of positive real numbers such that

$$\frac{f_n(x)}{c_n} \longrightarrow 0 \text{ for a.e. } x. \tag{3}$$

Hint: Pick c_n such that $m(\{x : |f_n(x)|/c_n > 1/n\}) < 2^{-n}$ and apply the Borel-Cantelli lemma.

6. Let $\mathcal{C}$ be the middle thirds Cantor set. Show that $x \in \mathcal{C}$ if and only if it has a ternary expansion of the form

$$x = \sum_{j=1}^{\infty} \frac{t_j}{3^j} \text{ where } t_j \in \{0, 2\}. \tag{4}$$

Note that the ternary expansion is not unique. Define $F : \mathcal{C} \to [0, 1]$ by letting

$$F(x) = \sum_{j=1}^{\infty} \frac{t_j/2}{2^j}. \tag{5}$$

Show that F is well defined and continuous on $\mathcal{C}$, and that $F(0) = 0$, and $F(1) = 1$, then show that F is surjective. Finally show that if (a, b) is a *maximal* open subset in $\mathcal{C}^c$, then $F(a) = F(b)$, and thereby extend $F : [0, 1] \to [0, 1]$, as a continuous map.

Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set. Hint: If $\mathcal{N} \subset [0, 1]$ is the non-measurable subset constructed in class, then consider $F^{-1}(\mathcal{N}) \cap \mathcal{C}$.

7. Let $\mathcal{N}$ be the non-measurable subset constructed in class. Show that any measurable set $E \subset \mathcal{N}$ has measure zero. Show that if G is a set with $m_*(G) > 0$, then G has a non-measurable subset.

8. In this problem we prove the following theorem: A bounded function f defined on an interval $J = [a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.

To prove this we use the following concept: For a bounded function f defined on a compact interval J and $0 < r$ let

$$\text{osc}(f, c, r) = \sup\{|f(x) - f(y)| : x, y \in J \cap (c - r, c + r)\}. \tag{6}$$

This is a non-decreasing function of r and therefore $\operatorname{osc}(f, c) = \lim_{r \to 0^+} \operatorname{osc}(f, c, r)$ is well defined; f is continuous at c if and only if $\operatorname{osc}(f, c) = 0$. To prove the statement above, prove the following assertions:

(a) For every $\epsilon > 0$ the set of points $A_\epsilon = \{x \in J : \operatorname{osc}(f, x) \geq \epsilon\}$ is compact.

(b) If the set of discontinuities of f has measure 0, then f is Riemann integrable. Hint: Cover A_ϵ by a finite collection of open intervals of length less than ϵ, then construct appropriate partitions of J on which to estimate the difference between the upper and lower Riemann sums.

(c) Conversely, if f is Riemann integrable on J, then its set of discontinuities has measure zero. Hint: The set of discontinuities of f is contained in $\cup_n A_{\frac{1}{n}}$. Construct a partition P so that

$$U(f, P) - L(f, P) \leq \frac{\epsilon}{n}. \tag{7}$$

Show that the total length of the intervals in P that intersect $A_{\frac{1}{n}}$ is at most ϵ.