AMCS 608

Problem set 1 due September 16, 2014

Dr. Epstein

Reading: There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. We can define polar coordinates in the complex plane by setting
$$x = r\cos\theta \quad y = r\sin\theta. \tag{1}$$
Show that in polar coordinates the Cauchy-Riemann equations take the form
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}. \tag{2}$$

2. Let $f(x, y) = \sqrt{|x||y|}$, for all $(x, y) \in \mathbb{R}^2$. Show that f satisfies the Cauchy-Riemann equation at $(0, 0)$, but does not have a complex derivative at this point.

3. Prove the following:

   (a) The power series $\sum_{n=1}^{\infty} nz^n$ converges for any z with $|z| < 1$, but does not converge for any point where $|z| = 1$.

   (b) The power series $\sum_{n=1}^{\infty} z^n/n^2$ converges for any z with $|z| \leq 1$.

   (c) The power series $\sum_{n=1}^{\infty} z^n/n$ converges for any z with $|z| \leq 1$, except $z = 1$.

4. Let $a, b, c \in \mathbb{C}$ under what conditions does the equation
$$az + b\bar{z} + c = 0 \tag{3}$$
have a unique solution? When does this equation define a line in the complex plane?

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Show that if f is analytic in $B_R(0)$ and $\operatorname{Re} f$, or $\operatorname{Im} f$ is a constant, then f is constant.

2. Suppose that g is an analytic function defined in $U \subset \mathbb{C}$ and f is an analytic function defined in an open set containing $g(U)$. Show that $f \circ g$ is an analytic function, using just the definition of complex derivative. Show that the chain rule is valid:
$$(f \circ g)'(w) = f'(g(w))g'(w). \tag{4}$$

3. Verify the chain rule for complex coordinates, e.g. $z, \bar{z}, w, \bar{w}$: Suppose that $f(z, \bar{z}) : U \to V$ and $g(w, \bar{w}) : V \to \mathbb{C}$, are differentiable in the real sense, i.e. as maps from subsets of $\mathbb{R}^2$ to $\mathbb{R}^2$. As noted in class we can think of f and g as functions of the variables $z, \bar{z}$. If we define $h = g \circ f$, then show that
$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}; \tag{5}$$
and
$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial \bar{z}}. \tag{6}$$
Recall that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Let f be analytic in a neighborhood of z. Show that the real derivative of f, $Df(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible at $z = x + iy$ if and only if $f'(z) \neq 0$.

4. Show that if f is a $\mathscr{C}^1$-function, then
$$df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}. \tag{7}$$
Show that any one form α can be written in the form
$$\alpha = adz + bd\bar{z}, \tag{8}$$
and that
$$d\alpha = (\partial_z b - \partial_{\bar{z}} a) dz \wedge d\bar{z}. \tag{9}$$
Show that the 1-form $\alpha = adz$ is closed ($d\alpha = 0$) if and only if a is an analytic function.

5. Suppose that f, defined in $D_1(0)$, is infinitely differentiable. Show that for each $n \in \mathbb{N}$ we have
$$f(z, \bar{z}) = \sum_{0 \leq j+k \leq n} \frac{\partial_z^j \partial_{\bar{z}}^k f(0, 0)}{j!k!} z^j \bar{z}^k + O(|z|^{n+1}). \tag{10}$$

6. If we define $\log z = \log r + i\theta$, at $z = r\cos\theta + ir\sin\theta$, then show that $\log z$ is an analytic function in the set

$$\{z \in \mathbb{C} : \ r \neq 0 \text{ and } -\pi < \theta < \pi\}.$$

Sketch this set. Use this to prove that if f is analytic in $B_R(0)$ and $|f(z)|$ is constant then f is constant.

7. Suppose that f is an analytic function defined in U and that $c : [0, 1] \to U$ is a C^1-curve. The composition $t \to f(c(t))$ can be thought of as a map h from $[0, 1]$ into $\mathbb{R}^2$. Show that the first derivative of this map can be computed using the chain rule:

$$Dh(t) = [\mathrm{Re}(f'(c(t))c'(t)), \mathrm{Im}(f'(c(t))c'(t))]. \tag{11}$$

Here we think of c as a complex valued function $c(t) = c_1(t) + ic_2(t)$, with c_1, c_2 real valued functions. Note: This does not follow from (4) as c is not an analytic map (or even defined in an open subset of $\mathbb{C}$).

8. Using the definition of the complex contour integral evaluate the following integrals over γ, the semi-circle $\{z : \ |z| = 1, \ \mathrm{Im}\, z > 0\}$, oriented from 1 to -1.

   (a) $\int\limits_\gamma \frac{dz}{z^n}$ where $n \in \mathbb{N}$.

   (b) $\int\limits_\gamma \bar{z}^n dz$, where $n \in \mathbb{Z}$.

   (c) $\int\limits_\gamma z^n \bar{z}^n dz$, where $n \in \mathbb{Z}$.

9. Suppose that $f(z, \bar{z})$ is a $\mathscr{C}^1$, complex valued function defined $D_1(0)$ that satisfies the partial differential equation

$$\partial_z f = 0. \tag{12}$$

Let Γ be a $\mathscr{C}^1$, simple closed in $D_1(0)$. What can you say about the value of

$$\oint\limits_\Gamma f(z, \bar{z}) d\bar{z}? \tag{13}$$

Give an example of non-constant function that satisfies (12).