Welcome to mini practice final exam 2, where the questions are made up and points don't matter. (However, it is probably helpful to take it seriously, including finishing on time, since it will help you learn what sections are your weakest.) Approximate time you should take to finish every question: 55-65 minutes. If you finish the first 10-11 questions in 50 minutes, time shouldn't be a problem on the real exam.

1. (18 points) Let $N = \dfrac{p+q}{p+r}$, where p, q, r are functions of u, v, w defined by $\begin{cases} p = u + vw, \\ q = v + wu, \\ r = w + uv. \end{cases}$

   Compute $\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$, at $u = 1, v = 2, w = 3$.

Solution: By the Chain Rule,

$$\frac{\partial N}{\partial \bullet} = \frac{\partial N}{\partial p}\frac{\partial p}{\partial \bullet} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial \bullet} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial \bullet}, \tag{1}$$

where $\bullet$ can be any choice of u, v, w. First, $p = 1 + 2 \cdot 3, q = 2 + 3 \cdot 1, r = 3 + 1 \cdot 2$, so

$$p = 7, q = 5, r = 5.$$

Now you should compute the partials of N with respect to p, q, r :

$$\frac{\partial N}{\partial p} = \frac{(p+r)-(p+q)}{(p+r)^2} = \frac{r-q}{(p+r)^2} = \frac{5-5}{\text{doesn't matter}} = 0,$$

$$\frac{\partial N}{\partial q} = \frac{1}{p+r} = \frac{1}{12}, \text{ and } \frac{\partial N}{\partial r} = \frac{-(p+q)}{(p+r)^2} = -\frac{1}{12}.$$

Therefore you don't need to find partial derivatives of p, since the first summand of (1) is always zero. The rest of the partials we need are

$$\frac{\partial q}{\partial u} = w = 3, \frac{\partial q}{\partial v} = 1, \frac{\partial q}{\partial w} = u = 1,$$

$$\frac{\partial r}{\partial u} = v = 2, \frac{\partial r}{\partial v} = u = 1, \frac{\partial r}{\partial w} = 1.$$

Finally, let's use the simplified form of (1), namely

$$\frac{\partial N}{\partial \bullet} = \frac{1}{12}\frac{\partial q}{\partial \bullet} - \frac{1}{12}\frac{\partial r}{\partial \bullet},$$

to conclude

$$\frac{\partial N}{\partial u} = \frac{1}{12}\frac{\partial q}{\partial u} - \frac{1}{12}\frac{\partial r}{\partial u} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12},$$

$$\frac{\partial N}{\partial v} = \frac{1}{12}\frac{\partial q}{\partial v} - \frac{1}{12}\frac{\partial r}{\partial v} = \frac{1}{12} - \frac{1}{12} = 0,$$

and

$$\frac{\partial N}{\partial w} = \frac{1}{12}\frac{\partial q}{\partial w} - \frac{1}{12}\frac{\partial r}{\partial w} = \frac{1}{12} - \frac{1}{12} = 0.$$

2. (6 points) Find the limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2}.$$

Solution: Since the numerator and the denominator are continuous everywhere, the limit exists everywhere except potentially at5 $(0,0)$. So we can just plug in, as long as we do not get $\frac{0}{0}$:

$$\lim_{(x,y)\to(1,2)} \frac{xy}{x^2+y^2} = \frac{2}{5}.$$

3. (6 points) Find the limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}.$$

Solution: When we try to plug in, we get $\frac{0}{0}$. Taking the paths $x=1$ or $y=2$, we see that the expression above simplifies to 0. Trying instead the line $y=x+1$ (***containing the point*** $(1,2)$***!!***), we get

$$\lim_{(x,y)\to(1,2)\text{ along } y=x+1} \frac{(x-1)(x-1)}{(x-1)^2+(x-1)^2} = \frac{1}{2}.$$

Since the limit is different along two different curves containing the limiting point $(1,2)$, the limit does not exist.

4. (6 points) Find the limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}.$$

Solution: When we try to plug in, we get $\frac{0}{0}$. However, notice the numerator is $(x^2+y^2)(x^2-y^2)$, so we can factor and cancel (x^2+y^2) :

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x^2-y^2 = 0.$$

Note that there were no squeeze limit questions here, but you might want to review that just in case, if these are too easy.

5. (9 points) Evaluate $\iint_D \frac{y}{1+x^2} dA$ where D is the region bounded by $y = \sqrt{x}, y = 0, x = 1$.

Solution: There are two ways to set up the integral:

$$\iint_D \frac{y}{1+x^2} dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dydx = \int_0^1 \int_{y^2}^1 \frac{y}{1+x^2} dxdy.$$

The second way is difficult (if you try both ways, you see the reason is that $\arctan(u)$ is more difficult to integrate than $\frac{1}{u}$). Using the $dydx$ integral,

$$\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dydx = \int_0^1 \frac{1}{2} \cdot \frac{y^2}{1+x^2}\bigg|_{y=0}^{y=\sqrt{x}} = \frac{1}{2} \int_0^1 \frac{x}{1+x^2} dx = \frac{1}{4} \ln(1+x^2)\Big|_0^1 = \frac{\ln(2)}{4}.$$

6. (9 points) Evaluate $\int_0^2 \int_{-\sqrt{4-y^2}}^0 \int_0^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2} dzdxdy$.

Solution: You should change to spherical coordinates. The region is a ball of radius 2 (known from each of the integrals), above the $xy-$plane (from the inner most integral, so $0 \leq \phi \leq \pi/2$); finally, if we draw the last two integrals we see that it is a circle of radius 2, but y must be positive and x must be negative, so the integration is only over the second quadrant $\pi/2 \leq \theta \leq \pi$. So the integral becomes

$$\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^2 (\rho \sin(\phi)\sin(\theta))(\rho)\rho^2 \sin(\phi) d\rho d\theta d\phi = \int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^2 \rho^4 \sin(\theta)\sin^2(\phi) d\rho d\theta d\phi$$

$$= \left(\int_0^{\pi/2} \sin^2(\phi) d\phi\right)\left(\int_{\pi/2}^{\pi} \sin(\theta) d\theta\right)\left(\int_0^2 \rho^4 d\rho\right) = \frac{\pi}{4} \cdot (-\cos(\pi) + \cos(\pi/2)) \cdot \frac{2^5}{5} = \frac{2^5\pi}{20}$$

since $\cos(\pi/2) = 0 = \cos(3\pi/2)$.

New section: TRUE OR FALSE

7. (5 points) If $PV = nRT$, where n, R are constants, then $\dfrac{\partial P}{\partial V} \cdot \dfrac{\partial V}{\partial T} \cdot \dfrac{\partial T}{\partial P} = 1$.

Solution: FALSE. Let $F(P, V, T) = PV - nRT = 0$. Using the chain rule, we derived previously that

$$\frac{\partial P}{\partial V} = -\frac{\partial F/\partial V}{\partial F/\partial P}.$$

Similarly for the other two partial derivatives. Therefore

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = \left(-\frac{\partial F/\partial V}{\partial F/\partial P}\right)\left(-\frac{\partial F/\partial T}{\partial F/\partial V}\right)\left(-\frac{\partial F/\partial P}{\partial F/\partial T}\right) = (-1)^3 = -1.$$

8. (5 points) $x^2 - 2y^2 = 4 - z^2$ is an equation of a hyperboloid of one sheet.

Solution: TRUE. $x^2 - 2y^2 + z^2 = 4$ is an equation of a hyperboloid of one sheet because (i) it is definitely the equation of some hyperboloid, and (ii) if we let $y = 0$, we see the cross section at zero is $x^2 + z^2 = 4$, a circle. Therefore this is an equation of a hyperboloid of one sheet.

9. (5 points) $2x^2 + 2y^2 = 4 - z^2$ is an equation of a hyperboloid of two sheets.

Solution: FALSE. Adding z^2 to both sides, we see $2x^2 + 2y^2 + z^2 = 4$ is the equation of an ellipsoid centered at the origin.

New section: MULTIPLE CHOICE

10. (7 points) Which of the two planes are parallel?

A. $x + y + z = 1$ and $x + y = 1$

B. $2x - 2y + 2z = 1$ and $2x + 2y + 2z = -1$

C. $x - y - z = 1$ and $z - x + y = 2$

D. $x + 2y + 3z = 2$ and $z + 2y + 3x = 4$

E. $-x - y - z = 0$ and $z = 0$

Solution: Two planes are parallel when their normal vectors are parallel (one is a multiple of the other). These normal vectors only differ by a negative sign: the first is $\langle 1, -1, -1\rangle$ and the second is $\langle -1, 1, 1\rangle$.

11. (7 points) The lines $\mathbf{r}_1(t) = \langle 1 - t, 1 + t, 2 + 3t\rangle$ and $r_2(s) = \langle 2s - 4, s, s + 3\rangle$ are

A. Parallel

B. Orthogonal

C. Intersecting

D. Skew

E. None of the above

Solution: The above lines intersect at $t = 1, s = 2$: solving the equations $1 - t = 2s - 4$ and $1 + t = s$, we see that $t = 1$ and $s = 2$ are the one places these lines *might* intersect. Checking the last coordinate, $2 + 3(1) = (2) + 3$, therefore the lines both equal $\langle 0, 2, 5\rangle$ at some time, so they intersect.

Skew means they do not intersect; the lines are not parallel because their directions are not multiples of each other ($\langle -1, 1, 3\rangle \neq C\langle 2, 1, 3\rangle$); the lines are not orthogonal because if we dot their tangent vectors (directions), we do not get zero: $\langle -1, 1, 3\rangle \cdot \langle 2, 1, 3\rangle = 8 \neq 0$. "None of the above" is impossible.

12. (7 points) The length of $\langle 2t^{3/2}, \cos(\pi t), \sin(\pi t)\rangle$ from $\langle 2, -1, 0\rangle$ to $\langle 16, 1, 0\rangle$ is

A. 14

B. $\sqrt{9t + \pi^2}$

C. $\frac{2}{27}(36 + \pi^2)^{3/2}$.

D. $\frac{1}{3}\Big((38 + \pi)^{3/2} - (14 + \pi^2)^{3/2}\Big)$

E. None of the above

Solution: The length is given by $\int_a^b |\mathbf{r}'(t)|dt$. In this case,

$$|\mathbf{r}'(t)| = \sqrt{(3\sqrt{t})^2 + (-\pi\sin(\pi t))^2 + (\pi\cos(\pi t))^2} = \sqrt{9t + \pi^2}.$$

Next, at $\langle 2, -1, 0\rangle$, we see $2 = 2t^{3/2}$ so $t = 1$, and therefore $a = 1$. Then at $\langle 16, 1, 0\rangle$ we have $t^{3/2} = 8$, so $t^3 = 64$, so $t = 4$, and therefore $b = 4$. The integral we must evaluate is

$$\int_1^4 \sqrt{9t + \pi^2}dt = (9t + \pi^2)^{3/2}(1/9)(2/3)\Big|_1^4 = \frac{2}{27}\Big((36 + \pi^2)^{3/2} - (9 + \pi^2)^{3/2}\Big).$$

13. (7 points) Find the linearization of xe^{xy} at $(2, 0)$.

A. $L(x, y) = x + 4y + 2$

B. $L(x, y) = x + 4y$

C. $L(x, y) = 4x + 2y + 1$

D. $L(x, y) = 4(x - 0) + 2(y - 0) + 2$

E. $L(x, y) = 2x$

Solution: If $f(x, y) = xe^{xy}$, then $f(2, 0) = 2$. Next,

$$\frac{\partial f}{\partial x}(2, 0) = e^{xy} + xye^{xy}\Big|_{(2,0)} = 1 \text{ and } \frac{\partial f}{\partial y}(2, 0) = x^2e^{xy}\Big|_{(2,0)} = 4.$$

The linear approximation is given by

$$L(x, y) = \frac{\partial f}{\partial x}(x - 2) + \frac{\partial f}{\partial y}(y - 0) + f(2, 0) = (x - 2) + 4(y - 0) + 2.$$