1. (12 points) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve composed from the straight line segments starting from $(1,0,0)$ going to $(0,1,0)$, then to $(0,0,1)$, and back to $(1,0,0)$, and $\mathbf{F}$ is the vector field $\langle x, xy, z\rangle$.

Solution: Since C is a closed curve, we may use Stokes's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

First, $\text{curl}(\mathbf{F}) = \langle 0, 0, y\rangle$. Next, the surface is the plane $x + y + z = 1$ (use the three points to find this equation, for example). Therefore a parametrization we can use from the integral is simple the rule $z = 1 - x - y = f(x, y)$, or $\mathbf{r}(x, y) = \langle x, y, 1 - x - y\rangle$. In this case,

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1\rangle = \langle 1, 1, 1\rangle.$$

This vector points up, which is the correct orientation since the curve goes counterclockwise when viewed from above (and therefore the $z-$component must be positive).

The region is given by $1 - x - y = 0$. Since we are in the first octant, we can use the limits $y = 1 - x$ and $y = 0$ for the inside integral. To get the next limits, set these limits equal ($x = 1$, and $x = 0$ is given since we are in the first octant). The integral becomes

$$\int_0^1 \int_0^{1-x} \langle 0, 0, y\rangle \cdot \langle 1, 1, 1\rangle dydx = \int_0^1 \int_0^{1-x} ydydx = \int_0^1 \frac{(1-x)^2}{2} dx = \frac{1}{6}.$$

2. (6 points) What if instead the curve C in the above problem was just the line segment from $(1,0,0)$ to $(0,1,0)$?

Solution: By the last question, since the curl of $\mathbf{F}$ is not zero, we know $\mathbf{F}$ is not conservative. Therefore we cannot use the Fundamental Theorem of Line Integrals for this question. We just have to parametrize and take the integral the first way we learned.

First, we must parametrize the curve; the simplest parametrization of this line segment is

$$\mathbf{r}(t) = \langle 1, 0, 0\rangle(1 - t) + \langle 0, 1, 0\rangle t, \text{ for } 0 \leq t \leq 1.$$

So $\mathbf{r}'(t) = \langle -1, 1, 0\rangle dt$. Also, $\mathbf{F}(\mathbf{r}(t)) = \langle (1-t), t(1-t), 0\rangle$. Finally,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -(1-t) + t(1-t) + 0 \;\; dt = \int_0^1 -t^2 + 2t - 1 \;\; dt = -\frac{1}{3}.$$

Note: this implies that if C is the curve from $(0,1,0)$ to $(0,0,1)$ to $(1,0,0)$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{6} - \frac{-1}{3} = \frac{1}{2}$.

3. (9 points) Set up, but do not evaluate, the integral representing the volume of the region bounded by $z = 1 + x$ and $z = \sqrt{2x^2 + y^2}$.

Solution: To set up the first integral, you should understand the first function is a plane and the second is a cone, so it seems that the plane cuts through the cone above a region containing the origin. At the origin, $z = 1 + x > \sqrt{2x^2 + y^2}$, so $z = 1 + x$ is the top function. Therefore the integral looks like

$$\iint_R \int_{\sqrt{2x^2+y^2}}^{1+x} dz dA.$$

To get the next limits, set the previous ones equal: $\sqrt{2x^2 + y^2} = 1+x$, so $2x^2+y^2 = (1+x)^2 = 1+2x+x^2$. The equation we are left with is

$$x^2 - 2x + y^2 = 1.$$

We need to complete the square: $x^2 - 2x = x^2 - 2x + 1 - 1 = (x - 1)^2 - 1$. So the above becomes

$$(x - 1)^2 + y^2 = 2.$$

This is a polar region, and in this case the change of variables is $(x - 1) = r\cos(\theta), y = r\sin(\theta)$, so $x = 1+r\cos(\theta), y = r\sin(\theta)$. The circle is radius $\sqrt{2}$ and it is a whole circle, so $0 \le r \le \sqrt{2}, 0 \le \theta \le 2\pi$. If we change to cylindrical, we also need to change our innermost integral. So the final answer is

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{\sqrt{2(1+r\cos(\theta))^2+r^2\sin^2(\theta)}}^{2+r\cos(\theta)} r dz dr d\theta.$$

4. (9 points) Evaluate $\iint_R e^{-x^2-y^2} dA$, where R is the region bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant ($x, y \ge 0$).

Solution: The region is the part of the disk of radius 2 in the first quadrant not containing the disk of radius 1. To translate this into polar coordinates (since it is a polar region), the last sentence implies $1 \le r \le 2$. Since we are in the first quadrant, $0 \le \theta \le \pi/2$. Finally, $r^2 = x^2 + y^2$, so the integral becomes

$$\int_0^{\pi/2} \int_1^2 e^{-r^2} r dr d\theta = \frac{\pi}{2} \int_1^2 e^{-r^2} r dr = -\frac{1}{2} \cdot \frac{\pi}{2} \int_1^2 e^{-r^2} (-2r dr) = -\frac{\pi}{4} e^{-r^2} \Big|_1^2 = \frac{\pi}{4} \left(\frac{1}{e} - \frac{1}{e^4} \right).$$

Pick the best answer.

5. (7 points) Let $f(x,y) = x^4 + y^4$. The point $f(0,0)$ is a

   A. Local maximum

   B. Local minimum

   C. Global maximum

   D. Global minimum

   E. Saddle point

   F. None of the above

   ~~G. Not enough information~~ (What do you mean? The whole function is given to you!)

Indeed all the information you get does not tell you anything, since the Hessian determinant is zero. However, $x^4 + y^4$ is zero at $(0,0)$ and positive everywhere else. Therefore $f(0,0)$ is a global minimum since it is the smallest value the function f attains. This is just a reminder of what minima and maxima really are, and that the tests we have aren't always sufficient.

6. (7 points) If $\mathbf{a} = \langle 0, 1, 1\rangle$, $|\mathbf{b}| = 4$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/4$, then $\mathbf{a} \cdot \mathbf{b}$ is

   A. 0

   B. $-2\sqrt{6}$

   C. $4\sqrt{2}$

   D. 4

   E. None of the above

$|\mathbf{a}| = \sqrt{2}$. We know $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\theta) = 4\sqrt{2}\frac{\sqrt{2}}{2} = 4$.

7. (7 points) A parametrization of the plane $x + 2y - 3z = 2$ (in some 2-D region) is given by

   A. $\langle t, 2t, -3t\rangle$, where $0 \le t \le 1$

   B. $\langle 3s\cos(t) - 2s\sin(t) + 2, s\sin(t), s\cos(t)\rangle$ where $0 \le s \le 1, 0 \le t \le 2\pi$.

   C. $\langle x, 2y, -3z\rangle = 2$, where x, y, z are any real numbers

   D. $\langle u, 2v, u + 2v - 2\rangle$, where $0 \le u \le 1, 0 \le v \le 1$

   E. None of the above

This is the only parametrization above with two variables (2-D region) that satisfies $x + 2y - 3z = 2$.

8. (7 points) If $\mathbf{r}(2) = \langle 0, 0, 4\rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$, then what is $\mathbf{r}(t)$?

A. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$

B. $\mathbf{r}(t) = \langle t, t^2, t^3 - 4\rangle$

C. $\mathbf{r}(t) = \langle t - 2, t^2 - 4, t^3 - 8\rangle$

D. $\mathbf{r}(t) = \langle t - 2, t^2 - 4, t^3 - 4\rangle$

E. $\mathbf{r}(t) =$ Charizard

Surprisingly, the answer is not E. With this kind of problem, you can just take the derivative of the answers and see that D is the only one that satisfies our conditions. Otherwise, you should use

$$\mathbf{r}(t) = \mathbf{r}(2) + \int_2^t \mathbf{r}'(u)du = \langle 0, 0, 4\rangle + \langle t, t^2, t^3\rangle - \langle 2, 4, 8\rangle = \langle t - 2, t^2 - 4, t^3 - 4\rangle.$$

9. (7 points) Find a good linear approximation of $f(x, y) = y + \sin(x/y)$ at $(.1, 2.9)$.

A. $L(.1, 2.9) = \frac{1}{3}(.1 - 0) + (2.9 - 3) + 3$

B. $L(.1, 2.9) = (.1 - 0) + (2.9 - 3)$

C. $L(.1, 2.9) = \frac{1}{3}(3 - 2.9)$

D. $L(.1, 2.9) = (x - 0) + \frac{1}{3}(y - 3)$

E. $L(.1, 2.9) = \frac{1}{3}2.9 + 3$

Note the updated answer A. First, $f(0, 3) = 3$. Also,

$$\frac{\partial f}{\partial x}(0,3) = \frac{1}{y}\cos(x/y)\bigg|_{(0,3)} = \frac{1}{3}, \text{ and } \frac{\partial f}{\partial y}(0,3) = 1 + \frac{-x}{y^2}\cos(x/y)\bigg|_{(0,3)} = 1.$$

The linear approximation is given by

$$L(x, y) = \frac{\partial f}{\partial x}(x - 0) + \frac{\partial f}{\partial y}(y - 3) + f(0, 0) = \frac{1}{3}(.1 - 0) + (2.9 - 3) + 3.$$

10. (7 points) Let $w = xe^{yz}$, where $x = 1 + 2t, y = \sin(t), z = \cos(t)$. What is $\frac{dw}{dt}$?

A. $\frac{dw}{dt} = 2e^{\sin(t)\cos(t)} + (1 + 2t)e^{\sin(t)\cos(t)}$

B. $\frac{dw}{dt} = \cos(t)e^{\sin(t)\cos(t)} + t^2e^{\sin(t)\cos(t)}$

C. $\frac{dw}{dt} = 2te^{\sin(t)\cos(t)} + \sin(t)e^{\sin(t)\cos(t)} + \cos(t)e^{\sin(t)\cos(t)}$

D. $\frac{dw}{dt} = e^{\sin(t)\cos(t)} + 2\sin(t)\cos(t)e^{\sin(t)\cos(t)} + \cos^2(t)e^{\sin(t)\cos(t)}$

E. $\frac{dw}{dt} = 2e^{\sin(t)\cos(t)} + (1 + 2t)\cos(2t)e^{\sin(t)\cos(t)}$

The chain rule tells us

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{yz}2 + xze^{yz}\cos(t) + xye^{yz}(-\sin(t)).$$

Plugging in x, y, z (and factoring out e^{yz} to save space) we get

$$\frac{dw}{dt} = e^{\sin(t)\cos(t)}(2 + (1+2t)\cos^2(t) - (1+2t)\sin^2(t)) = e^{\sin(t)\cos(t)}(2 + (1+2t)[\cos^2(t) - \sin^2(t)]).$$

The cosine double angle formula tells us $\cos(2t) = \cos^2(t) - \sin^2(t)$ (on the cheat sheet on the exam, but good to know without it).