I work in the area of cluster algebras and its relations to higher Teichmüller spaces. Cluster algebras combines aspects of representation theory, combinatorics, geometry, and topology. I would first like to give a little context for the subject. Cluster algebras were invented by Fomin and Zelevinsky [9] in 2000 in a project whose aim was to create a combinatorial model for approaching the results by Lusztig concerning total positivity in algebraic groups [17], as well as look at canonical bases in quatum groups [16]. These initial goals are still far away and in many ways this is still the underlying motivation for work involving the theory of cluster algebras. Great strides have been made in the development of cluster algebras linking them with a wide breath of mathematical subjects including Poisson geometry, integrable systems, higher Teichmuüller spaces, combinatorial polyhedra, the representation theory of quivers and finite-dimensional algebras, commutative and non-commutative algebraic geometry, and in particular Calabi-Yau algebras and Donaldson-Thomas invariants. In addition to their connection to other areas of mathematics; cluster algebras have interdisciplinary connections. Some of my results have direct connections to particle physics and gauge theories. This connection will be explained further in the second section.

The first section gives a brief introduction to cluster algebras intended to be accessible to all mathematicians. The second section section outlines the work that I have done in the field of cluster algebras. The final section will address some research that I am currently working on, including a few projects that I think would be very accessible projects for undergraduate research.

Brief Introduction to Cluster Algebras From Quivers

A cluster algebra, $\mathcal{A}$, of rank n is a subring of the field of rational functions on n variables. We call this field of rational functions the ambient field, $\mathcal{F}$. Where cluster algebras differ from a standard commutative ring structure is that they are generally not given by a presentation of generators and relations. Instead, we describe a cluster algebra in terms of *seeds* and *quivers*.

You start by taking a directed graph with no loops or two cycles. In cluster algebras we call this a quiver, denoted Q. We then label each vertex of the quiver with a variable. The set of labels used in the quiver is the seed associated to the quiver, denoted Σ. The pair is often denoted (Q, Σ).

Example 1. In this example $\Sigma = \{x_1, x_2, x_3\}$.

$$x_1 \longrightarrow x_2 \longrightarrow x_3$$

We then define a process called mutation which will create a new pair (Q', Σ') from our previous pair. We define mutation in the following way.

Mutating the Quiver

The **mutation** of a quiver Q at a vertex k is denoted μ_k, and produces a new quiver $\mu_k(Q)$. The vertices of $\mu_k(Q)$ are the same vertices from Q. The arrows of the new quiver are obtained by performing the following 3 steps:

1. For every 2-path $i \to k \to j$, adjoin a new arrow $i \to j$.
2. Reverse the direction of all arrows incident to k.
3. Delete any 2-cycles created during the first two steps.

Mutating the Seed

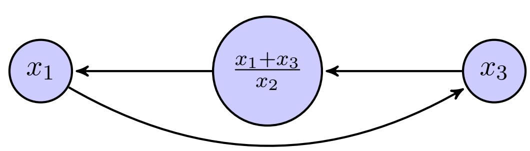
Now we've established how we get the mutated quiver, but we still need a way to get the new labels for each vertex. The vertices are now labeled as follows:

1. If a vertex is not the vertex which we mutated at, then its label remains the same.
2. For the mutated vertex we replace the label with

$$\frac{\prod\limits_{i \to k} x_i + \prod\limits_{i \leftarrow k} x_i}{x_k}$$

where the two products in the fraction are the product of the labels of the arrows going into the vertex and the product of the labels of the arrows going out of the vertex.

Example 2. Lets look at mutating our previous example at the second vertex, $\mu_2(Q)$. You can see the new quiver and we see that the new seed is the set $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$.



The cluster algebra, $\mathcal{A}$, is the algebra which is generated by all the seeds we get by performing any finite number of mutations. In other words the algebra $\mathcal{A}$ is generated by the set $\bigcup\limits_j \Sigma_j$, where Σ_j is any seed we can reach by mutation. Essentially the end result is a subring that is somewhere between the ring of polynomials in n variables and the ring of rational functions in n variables.

Past Accomplishments: Cluster Algebras from Surfaces and Green Sequences

Some of my results have focused on cluster algebras from surfaces. In general a cluster algebra can be constructed from any orientable surface by looking at the possible triangulations of that surface. This construction is introduced by Gekhtman, Shapiro, and Vainshtein in [10] and in a more general setting by Fock and Goncharov in [5].

This is a particularly important class of cluster algebras because they govern the cluster structure of Teichmüller spaces as well as decorated Teichmüller spaces. Another interesting area where cluster algebras from surfaces have a direct application is in physics. In particle theory there are objects called gauge theories, and most gauge theories of interest arise from looking at particle movement on a specific surface. In general the cluster structure associated to that surface allows us to compute the full BPS state of the gauge theory.

The results that I have completed on maximal green sequences, have the before mentioned ramifications in particle physics. The idea of maximal green sequences of cluster mutations was introduced by Keller in [12] and has a direct impact on algebraic geometry, Lie theory, in addition to its particle physics implications. Keller explored important applications of this notion, by utilizing it in the explicit computation of noncommutative Donaldson-Thomas invariants of triangulated categories which were introduced by Kontsevich and Soibelman in [13]. Very recently this notion also played a key role in the Gross-Hacking-Keel-Kontsevich [11] proof of the full Fock-Goncharov conjecture for large classes of cluster algebras. This conjecture produces canonical bases for these cluster algebras. For example, by considering the open double Bruhat cell, U, in the basic affine space Y we obtain a canonical basis of each irreducible representation of SL_r. In particle physics a maximal green sequence gives a method of computing the spectra of BPS states in various gauge theories [1]. In essence, the gauge theory can be modeled by a cluster quiver. Since each gauge theory is associated to a surface, the existence of a maximal green sequence for any corresponding cluster algebra gives a very tangible way of computing the BPS states.

The problem of existence of maximal green sequences of cluster mutations is difficult due to the iterative nature of the choices of mutations. This means that exhaustive methods are not always effective when searching for a maximal green sequence. My work has proven the existence of maximal green sequences for multiple infinite families of cluster algebras that are associated to surfaces. In general these infinite families were fairly interesting because they were the only remaining surfaces where we were unsure whether a cluster algebra with a maximal green sequence could be associated to the surface. Below are two of my key results:

Theorem 1 ([3], B.)**.** *Let Q_n^p be the quiver obtained from our triangulation of a genus n surface with no boundary. The quiver Q_n^2 has a maximal green sequence. (In my work I explicitly give the green sequence.)*

Theorem 2 ([4], B. , Mills)**.** *Let Q_n^p be the quiver obtained from our triangulation of a genus n surface with no boundary and $p \geq 3$ punctures. Then Q_n^p has a maximal green sequence. (In our work we explicitly give the green sequence.)*

Future Research Objectives

- There has been a recent development in mirror symmetry and tropical geometry involving the use of a technical tool called a scattering diagram. A scattering diagram is a collection of "scattering walls" each attached with a "scattering term". A scattering diagram can be computed for any cluster algebra. At the moment the way a scattering diagram is constructed uses an iterative approach developed in [11]. In general scattering diagrams give us a ground breaking new way to look at the theory of cluster algebras. They were the primary tool that allowed Gross, Hacking, Keel, and Kontsevich to prove the canonical bases conjecture mention above, which was a large step in accomplishing the initial goals set out by Fomin and Zelevinsky. My belief is that there is a way to look at the Teichmüller theory of the surface and construct the scattering diagram directly. This would mean that we could read off the existence of various walls without having the complete picture of the scattering diagram. My collaborator and I have already gotten some results on this using the theory of laminations developed by William Thurston in the 1980's. This approach would be non-iterative, and and give us a way to seek out particular wanted or unwanted walls and study them regardless of where they occur in the iterative process. The ramifications of this would be very impactful, because we would then be able to answer questions about cluster structures without needing complete knowledge first. I'll give a quick example of this importance. A scattering diagram for a given cluster algebra is a collection of walls in $\mathbb{R}^n$ where n is the rank of the cluster algebra. In order compute walls in the iterative process we must consider all paths from one chamber to another. This is possible, but difficult when you get to higher dimensions. Our method reduces this process to a question of topology on surfaces, which is much easier to manipulate. I believe this will lead to many cluster algebra results that are currently proving allusive using more standard techniques. One early example where scattering diagrams have already shed light on an otherwise difficult question is in the work of Muller [18]. Muller uses scattering diagrams to show that the existence of a maximal green sequence is not a mutation equivalent property.

- In a paper by Fomin, Shapiro, and Thurston [8] in 2007 the ground work was laid out for most of the construction of cluster algebras associated to triangulated surfaces. In the years since the term cluster algebras from surfaces has become regularly used in the community. In some was this colloquialism is a misnomer, because there are in fact multiple ways in which one can construct a cluster algebra from a marked surface. The special linear group $SL(V)$ of a finite-dimensional complex vector space V with a volume form ω, has a natural action on the coordinate ring $\mathbb{C}[(V^*)a \times V^b \times (SL(V)^c]$. Fomin and Pylyavskyy [6] [7] showed that the ring of invariants under this action carries with it a cluster structure that can be modeled combinatorially by tensor diagrams on surfaces. This cluster algebra is denoted $R_{a,b,c}(V)$. By looking at the Kuperberg diagrammatic calculus [14] [15] for tensor diagrams they seeked to try and understand what the seeds and cluster monomials are in these cluster algebras. A special class of tensor diagrams called irreducible webs were shown in [7] to correlate to a spanning set for the cluster algebra $R_{a,b,c}(V)$ when $\dim V = 3$, but it still remains a open conjecture to prove that this is in fact a basis as well. I believe there is a way to prove the Fomin-Pylyavskyy conjecture, by embedding the cluster algebra into a non-commutative algebra with a Poisson structure. We can then use various geometric techniques to make a dimension argument to show that this set is linearly independent. One ramification of proving the conjecture would be that for the algebra $R_{a,b,c}(V)$ the uppercluster algebra would equal the cluster algebra. In addition to the Fomin-Pylyavskyy conjecture, there are many open areas of research in the cases where $\dim(V) > 3$. This is largely because the Kuperberg calculus becomes more difficult to manipulate. In general tensor diagrams are k-regular bipartite graphs drawn on a surface where k is the dimesion of V, therefore the local relations grow in complexity as k increases. Another point of difficulty is that in higher dimensions it is unknown what set of diagrams would play the role of the irreducible webs. It is my belief that with an embedding into a larger algebra we might be able to develop a way of understanding the cluster structure of $R_{a,b,c}(V)$ for higher dimensional V. These algebras are intersting objects in representation theory as well as invariant theory, and results about there cluster structure would have an impact on these fields.

- There are many families of cluster algebras where the existence of maximal green sequences is still unknown. I intend to look further into a wide assortment of families where this question remains open. In general, many of these families are very accessible for potential undergraduate research. I believe that projects involving maximal green sequences and these cluster algebras would make great REU research programs and undergraduate thesis projects. They can be approached in a combinatorial way, which doesn't require much understanding of the algebraic and topological underlying structures. Students can have an integral role in proving results that have a wide range of mathematical implications, while not being overwhelmed by a large amount of background necessity. This helps peak mathematical interest and give the students a research building block to use if they choose to continue with a career in mathematics.

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