1. Using the axioms for Real Numbers, complete the following proof that \( \mathbb{R}^+ = \{ x \in \mathbb{R} \mid 0 < x \} \)

Proof that \( \subseteq \): Let \( x \in \mathbb{R}^+ \). We want to compute \( x - 0 \) to see if it is a member of positive real numbers as well.

Since by axiom 8.2 we have \( 0 + 0 = 0 \). We claim that additive inverses are unique*, hence by axiom 8.4 \( -0 = 0 \) and hence \( x - 0 = x + 0 = x \).

Therefore we get \( x - 0 = x \in \mathbb{R}^+ \) which by definition means \( 0 < x \).

Proof that \( \supset \): Let \( y \in \{ x \in \mathbb{R} \mid 0 < x \} \), we want to show \( y \in \mathbb{R}^+ \).

[Also prove *: if \( b_1 \) and \( b_2 \) are additive inverses of \( a \), then \( b_1 = b_2 \).]

2. The product of a positive real number with a negative real number is a negative real number. (Hint: let \( x \) be positive, \( y \) be negative, then \( (-y) \) is positive. Assume that \( x \cdot y \) is positive. Show that \( x \cdot y + x \cdot (-y) = 0 \) and hence we get a contradiction since the sum of two positive numbers is positive but \( 0 \notin \mathbb{R}^+ \).

3. Using the axioms for Real Numbers, complete the proof that \( 0 < 2^{-1} \).

Assume the contrary: assume that \( 2^{-1} \) is not greater than zero, which means \( 2^{-1} - 0 \notin \mathbb{R}^+ \).

Then by axiom 8.26(iv), \( -2^{-1} \in \mathbb{R}^+ \).

Since \( 2 = 1 + 1, 0 < 2 \) by axiom 8.26(i).

By axiom 8.26(ii) we get \( 2 \cdot (-2^{-1}) \in \mathbb{R}^+ \),

... [show that this product is the additive inverse of 1, hence not positive, which will be the contradiction]

4. Using the axioms for Real Numbers, prove that if \( x \) and \( y \) are positive real numbers, and \( x < y \), then there is a real number \( z \) such that \( x < z \) and \( z < y \). (Recall: \( x < y \) means \( y - x \in \mathbb{R}^+ \), and that \( 0 < 2^{-1} < 1 \))

5. Show that the least upper bound of \( A = \{ -1/k \mid k \in \mathbb{Z}^+ \} \) is zero. (First show that 0 is an upper bound, then show that it is smaller than any other upper bound, for example by showing that any number smaller than zero is not an upper bound)

6. Complete the following proof of: If a sequence \( (a_n) \) is strictly increasing and has a limit \( L \), then \( \lim(a_n) = \sup \{ a_n \mid n \in \mathbb{N}^+ \} \)

Let \( M = \sup \{ a_n \mid n \in \mathbb{N}^+ \} \). Since \( (a_n) \) has a limit, it is bounded, so by axiom 8.52, \( M \) exists.

Proof that \( L \) is an upper bound: Assume not, then there is a \( k \in \mathbb{N}^+ \) such that \( a_k > L \). Let \( \epsilon = a_k - L \). Then \( \epsilon > 0 \). Since \( (a_n) \) is strictly increasing, \( a_m > a_n \) for all \( m > n \), hence \( a_m - L > a_n - L = \epsilon \). Therefore we can’t find an \( N \in \mathbb{N}^+ \) that satisfies the requirements of the limit criterion, so \( L \) is not the limit, which is a contradiction.

Proof that \( L \) is the smallest upper bound: Assume \( S \) is a smaller upper bound, then \( L - S > 0 \). Let \( \epsilon = L - S \). Since \( L \) is the limit, there is an \( N \in \mathbb{N}^+ \) such that for all \( n > N, a_n \in (L - \epsilon, L + \epsilon) \). ... [Now pick \( a_{N+1} \) and show that it is greater than \( S \), hence \( S \) is not an upper bound.]