PROBLEM 1. Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

(a) Prove the following statement.

(*) If $f$ is surjective and $g$ is surjective, then the composition $g \circ f : A \rightarrow C$ is surjective.

(b) Identify the hypothesis and conclusion of statement (*), above.

(c) State the inverse, contrapositive and converse of statement (*). Determine whether each of these is true or false. For each true statement, provide a short proof; for each false statement, provide a counterexample.

PROBLEM 2. Prove the statements appearing in (a)-(c), and answer the prompt in (d). The symbol $\cong$ denotes bijective correspondence.

(a) For all sets $A$ and $B$, if $A \cong B$, then $B \cong A$.

(b) Suppose $A$ is a set. Then $A \cong A$.

(c) For all sets $A$, $B$ and $C$, if $A \cong B$ and $B \cong C$, then $A \cong C$. *Hint: Use problem 1 above, and one of the homework or essay problems.*

(d) State the negation of each of the statements (a)-(c) above. Determine if the negation is true or false. Provide a counterexample for any false statement.

PROBLEM 3. Let $E$ denote the set of even integers.

(a) Use a picture to illustrate a bijection between $\mathbb{N}$ and $E \times E$.

(b) Use a picture to illustrate a bijection between $\mathbb{Z}$ and $E \times E$.

PROBLEM 4.

(a) Find a set $S \subseteq \mathbb{R}$ such that the function

$$f : [0, \infty) \longrightarrow S$$

$$x \longmapsto \frac{1}{1 + x^2}$$

is surjective.
(b) Let \( f \) and \( S \) be as in part (a). Prove that \( f \) is injective.

(c) Let \( S \) be as in part (a). Suppose \( T \) is a set which is strictly larger than \( S \); that is, \( S \subseteq T \), but \( S \neq T \). Explain why the function

\[
g : [0, \infty) \rightarrow T
\]

\[
x \mapsto \frac{1}{1 + x^2}
\]

is not surjective.

(d) Let \( S \) be as in part (a). Suppose \( R \) is a set which is strictly smaller than \( S \); that is, \( R \subseteq S \), but \( R \neq S \). Explain why there is no function of the form

\[
e : [0, \infty) \rightarrow R
\]

\[
x \mapsto \frac{1}{1 + x^2}
\]

PROBLEM 5. Negate the following.

(a) \( \forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \) such that \( m \cdot n = 1 \).

(b) \( \exists x \in \mathbb{Q} \) such that \( \forall y \in \mathbb{Q}, \ x \cdot y = y \).

Rewrite the statements in (a) and (b) without the use of quantifiers and state if it is a true or a false statement. If it is a true statement, prove it. If it is a false statement, provide a counterexample.

PROBLEM 6. Construct a truth table to show that the contrapositive of \( A \Rightarrow B \) is equivalent to \( A \Rightarrow B \).

PROBLEM 7. Prove the following statement.

\[
\forall a \in \mathbb{R} \exists! x \in \mathbb{R}, \text{ such that } 3x - 1 = a.
\]

PROBLEM 8. Let \( E \) denote the set of even integers and \( A \) be the following statement.

\[
A : "x \in E \Rightarrow \exists k \in \mathbb{Z} \text{ such that } x = 2k"
\]

(a) Write the inverse of statement \( A \).

(b) Write the converse of statement \( A \).

(c) Write the contrapositive of statement \( A \).

(d) Is statement \( A \) true? What about its converse? In this case, how would you restate it using necessary/ sufficient/ necessary and sufficient?
(e) Which of the statements in parts (a), (b), or (c) is equivalent to the original statement in general (no matter what $A$ is)?

PROBLEM 9. Let $A = \{x \in \mathbb{Z}|x = 6k, k \in \mathbb{Z}\}$, $B = \{x \in \mathbb{Z}|x = 2k, k \in \mathbb{Z}\}$, $C = \{x \in \mathbb{Z}|x = 3k, k \in \mathbb{Z}\}$. Prove the following statement.

$$x \in A \iff \exists y \in B \text{ and } \exists z \in C \text{ such that } x = yz.$$