Research Statement

David L. Duncan

My research is in gauge theory and symplectic geometry. Each of these fields provides powerful invariants for 3- and 4-manifolds by counting solutions to certain PDEs. Much of my work analyzes these solution sets from various perspectives. In brief, my interests and activities include the following.

- **Atiyah-Floer conjectures**: An Atiyah-Floer conjecture is a formal statement to the effect that invariants in gauge theory can be equivalently described by invariants in symplectic geometry. Usually the precise formulation depends on the underlying manifold, or class of manifolds. These conjectures typically arise through heuristic adiabatic limits and reflect deep relationships between the defining equations. Much of my work seeks to make these heuristics precise.

- **Curves with boundary degenerations**: One pervasive difficulty with the various Atiyah-Floer conjectures (and a partial explanation for why there is more than one conjecture) is that it is not always clear how to define the relevant symplectic invariants in a coherent way. However, my analysis of these conjectures suggests a definition of these symplectic invariants using holomorphic curves with a specific type of boundary degeneration. Work in this direction is joint C. Woodward.

- **General structure groups**: In gauge theory one typically fixes a Lie group $G$, called the structure group. Traditionally one chooses $G = \text{SO}(3), \text{SU}(2)$ or $\text{SL}(2, \mathbb{C})$, since these choices tend to allow one to avoid singularities in moduli spaces. However, there have been numerous technological advancements in recent years that are making it possible to define gauge theoretic invariants with more general structure groups, and it is conceivable that these new invariants detect more refined data. I have made contributions in this direction through Chern-Simons computations for arbitrary (compact, connected) Lie groups $G$, and by defining an instanton Floer theory for $G = \text{PU}(r)$, with $r \geq 2$.

- **Quantum cup product in Heegaard Floer theory**: Heegaard Floer theory is an example of a symplectic-geometric invariant for 3-manifolds that has proven to be quite successful in that it is a powerful invariant that is often computable. On the other hand, the quantum cup product is a general construction in symplectic geometry that can detect information that is difficult to observe by more traditional means. In a joint project with M. Hedden and J. Baldwin, I am exploring the quantum cup product in Heegaard Floer theory, its usefulness for detecting topological data of the underlying 3-manifold, and its amenability to computation.

In the sections, below I elaborate on some of my contributions in these areas, and explain various future directions. Throughout, $Z$ will be an oriented 4-manifold with cylindrical ends, and $P \to Z$ will be a principal $G$-bundle, where $G$ is a compact, connected Lie group. The invariants in which I am most interested are the Donaldson invariants in gauge theory and certain holomorphic curve invariants in symplectic geometry. When these are defined, they depend only on the diffeomorphism type of $Z$, the Lie group $G$, and the topological type of the bundle $P$.

More explicitly, the **Donaldson invariant** of $(Z, P, G)$ is obtained by counting isolated **instantons**; that is, solutions $A$ to the **instanton equation**, 

$$ (1 + *)F_A = 0, $$

where $A$ is a connection on $P$ with curvature $F_A$, and $*$ is the Hodge star. Strictly speaking, one needs to choose a Riemannian metric on $Z$ for $*$ to be meaningful, however the resulting invariant is independent of the choice of metric; see Donaldson [6].

The relevant symplectic-geometric objects were introduced by Wehrheim and Woodward [37, 36], and Wehrheim [33]. For general $Z$, these arise by counting certain holomorphic quilts. To avoid digressing too far into the language of quilts, I will mostly restrict attention to the case where $Z = S \times \Sigma$ is a product of surfaces, with $\Sigma$ closed and connected; however, I will return to the general case in Section 2. For this restricted class of 4-manifolds the more familiar language of holomorphic **curves** suffices. Then to obtain the relevant symplectic invariant of $(Z, P, G)$, one counts isolated solutions $u : S \to M(\Sigma)$ to the **holomorphic curve equation**

$$ \overline{\partial} u = 0, \tag{1} $$

where $M(\Sigma)$ is a certain representation variety associated to the flat connections on $\Sigma$, and is defined using the data of $P, G$. The operator $\overline{\partial}$ is the standard Cauchy-Riemann operator. Its definition requires the choice of a complex structure $\mathcal{I}$ on $M(\Sigma)$ that is analogous to the choice of metric in the instanton case above.
Remark 1. (a) One obtains invariants of 3-manifolds \( Y \) by considering \( Z := \mathbb{R} \times Y \).

(b) There are two technical considerations one often imposes that ensure the above determines well-defined invariants. First, one assumes the bundle \( P \) has been chosen appropriately so as to avoid reducible connections. There is often a way to do this, though not always. Second, one often works with a perturbed version of the instanton and holomorphic curve equations so that the associated moduli spaces of solutions is smooth.

1 Atiyah-Floer conjectures

Let \( Z = S \times \Sigma \) be as above and take \( G = \text{SO}(3) \). The relevant Atiyah-Floer conjecture for \((Z, P, G)\) says that the Donaldson invariant equals the holomorphic curve invariant, and I will refer to this as the product AF conjecture. To describe why this is expected, it is useful to mention that any map \( u : S \to M(\Sigma) \) can be lifted to a connection \( A_0 \) on \( P \), and \( A_0 \) is unique up to bundle automorphism on \( P \). If \( u \) is a holomorphic curve, then I refer to any lift \( A_0 \) as a holomorphic representative. The usefulness of this is that we can view both holomorphic curves and instantons as elements in the same space.

The product AF conjecture arises due to the following heuristic, dating back to Atiyah [1], suggesting that the space of instantons is very close to the space of holomorphic representatives: The product structure of \( Z = S \times \Sigma \) induces a splitting of the instanton equation into two equations. The first of these equations is essentially the holomorphic curve equation (1), and the second equates a certain curvature on \( 0 \) with a certain curvature on \( \Sigma \). Recalling that the Donaldson invariants are independent of the choice of metric, one chooses a metric on \( Z \) of the form \( g_S + \epsilon^2 g_\Sigma \), where \( g_S \) (resp. \( g_\Sigma \)) is a metric on \( S \) (resp. \( \Sigma \)) and \( \epsilon > 0 \) is a small parameter. One can check that the holomorphic curve equation is independent of \( \epsilon \), but the second curvature equation forces the curvature component on \( \Sigma \) to be on the order of \( \epsilon^2 \). In the heuristic limit as \( \epsilon \) decreases to zero, one therefore recovers exactly the equations for a holomorphic representative.

A natural approach for proving the conjecture is therefore to show that, for small \( \epsilon \), every instanton is close to a unique holomorphic representative, and vice-versa. Toward this end, I have defined two maps

\[
\{\text{instantons}\}_k \xrightarrow{\mathcal{N}} \{\text{holomorphic representatives}\}_k \quad 0 \leq k \leq 3. \tag{2}
\]

The subscript \( k \) indicates that we are restricting attention to those instantons and holomorphic representatives that are described by moduli spaces of dimension \( k \); the dimension relevant for the conjecture is \( k = 0 \). The map \( \mathcal{N} \) was defined in [9] using the complex gauge action on the surface \( \Sigma \). The main theorem is the following, which will appear in [14].

**Theorem 1** (D. Duncan-J. McNamara[1]). The map \( \mathcal{N} \) is a smooth embedding.

The product AF conjecture would follow if we can show that \( \mathcal{N} \) is a surjection.

This is where the map \( \mathcal{M} \) enters the picture. I expect this to be an approximate inverse of \( \mathcal{N} \), and its definition relies on two observations. The first is that instantons are the global minimizers of the Yang-Mills functional \( \mathcal{Y}\mathcal{M} \), which is a real-valued function on the space of connections. The second is that the Yang-Mills value of a holomorphic representative is very near the global minimum:

\[
\mathcal{Y}\mathcal{M}(\text{holomorphic representative}) - \mathcal{Y}\mathcal{M}(\text{instanton}) \leq \epsilon^2 C.
\]

The idea is to use the gradient flow of the Yang-Mills functional to define \( \mathcal{M} \). To carry this out requires two ingredients. The first is an existence and uniqueness statement for this flow equation. Since \( Z \) is a manifold with cylindrical ends, we restrict attention to the space of connections that converge to a fixed (necessarily flat) connection \( a \) on the 3-manifold at infinity. Associated with \( a \) is an integer \( k(a) \in \mathbb{Z} \) (this is the \( k \) in [2]).

**Theorem 2** (M. Struwe, A. Schlatter, A. Waldron, D. Duncan). Let \((Z, g)\) be an oriented Riemannian manifold with cylindrical ends, and \( P \to Z \) an SO(3)-bundle with the property that all instantons are irreducible. Fix a small perturbation of the instanton equation so that all instantons are regular. Then there is some \( \eta > 0 \), depending on this data, so that the following holds: Fix a connection \( a \) on the 3-manifold at infinity with \( 0 \leq k(a) \leq 3 \). Suppose \( A_0 \) is a

---

[1] Jake McNamara is an exceptionally talented undergraduate at Harvard. He recently came on board with this project as an REU student jointly supervised by Chris Woodward and me.
connection on $P$ that converges to $a$ and with the property that $\mathcal{Y}\mathcal{M}(A_0)$ is within $\eta$ of the absolute minimum value of $\mathcal{Y}\mathcal{M}$. Then the gradient flow $\mathcal{Y}\mathcal{M}$, starting at $A_0$, has a unique solution $A_\tau$ for all time $\tau \in [0, \infty)$. Moreover, the solution $A_\tau$ converges, as $\tau$ approaches infinity, exponentially in $L^2(Z)$ to a unique instanton.

My contribution, to appear in [13], has been to extend this to the case of cylindrical ends and to include a perturbation; the case where $Z$ is closed and has no cylindrical ends has been known for some time and appears, in various forms, in [31], [30], [29], [32], and [17].

To describe the second ingredient in the definition of $\mathcal{M}$, let $\eta > 0$ be the constant from Theorem 2 for $Z = S \times \Sigma$ with the metric $g = g_S + e^2 g_\Sigma$ from above. Then $\eta$ certainly depends on the metric, and so it is conceivable that it depends on the parameter $\epsilon$; write $\eta(\epsilon)$ to denote this dependence. The following result will appear in [13], and relies on the compactness theorem [9].

**Theorem 3** (D. Duncan). There is some $\eta_0 > 0$ so that $\inf_{0 < \epsilon \leq 1} \eta(\epsilon) \geq \eta_0$.

Combining Theorems 3 and 2 shows that, when $\epsilon$ is sufficiently small, the Yang-Mills heat flow starting at a holomorphic representative $A_0$ converges to a unique instanton $\mathcal{M}(A_0)$, which is the definition of the map $\mathcal{M}$. It follows that proving the product AF conjecture reduces to proving the following:

**Conjecture 4.** The map $\mathcal{M}$ is injective when $\epsilon > 0$ is small.

Analysis of the Yang-Mills heat flow shows that this conjecture further reduces to establishing a uniform regularity estimate for small energy connections along the Yang-Mills gradient trajectories emanating from a holomorphic representative. I am currently pursuing several ideas for carrying this out.

As mentioned above, we restricted to the product case $S \times \Sigma$ to simplify the notation. However, with the technology of D. Gay and R. Kirby’s Morse 2-functions [19] in low-dimensional topology, and K. Wehrheim and C. Woodward’s pseudoholomorphic quilts [37] in symplectic geometry, the above discussion has extensions to nearly all oriented 4-manifolds that are either closed or have cylindrical ends; see Wehrheim- Woodward [36] and Wehrheim [33]. Moreover, the discussion above carries over to this more general case and it is very likely that techniques used to resolve Conjecture 4 in the product manifold case will apply in the much more general setting.

Atiyah-Floer conjectures have a fairly long history. The original Atiyah-Floer conjecture addressed the case $Z = \mathbb{R} \times Y$, where $Y$ is a homology 3-sphere, and $G = SU(2)$; this is described in [1]. Unfortunately, the relevant symplectic invariant is ill-defined due to issues with reducible connections (the topological conditions on $Y$ prohibit the existence of a suitable bundle; see Remark 1(b)). The first well-defined variant of the conjecture was considered by S. Dostoglou and D. Salamon, and addresses $Z = \mathbb{R} \times Y$ where $Y$ is a mapping torus. They resolved the conjecture in this case [7], and their proof serves in many ways as the model for the more general constructions described here. More recently, D. Salamon and K. Wehrheim describe a program [26] [27] to resolve the original Atiyah-Floer conjecture for homology 3-spheres; see also [33]. M. Lipyanskiy has also announced a program for proving these types of conjectures [23].

## 2 Quilts with boundary degenerations

As described in Section 1, the relevant symplectic invariant addressing the most general class of 4-manifolds involves holomorphic quilts. A **holomorphic quilt** is a finite collection of holomorphic curves $u_i : S_i \to M_i$ with certain prescribed boundary conditions. That is, each $M_i$ has a complex structure $J_i$, and $u_i$ is holomorphic with respect to this structure. Often the complex structure $J_i$ is allowed to depend on the domain $S_i$ as well.

In this general 4-manifold setting, one can revisit the construction of the map $\mathcal{N}$ from [2] and find that the natural codomain is the space of connections that represent holomorphic quilts (as expected) but with the property that the complex structures $J_i(s)$ diverge, in a very precise sense, as $s \in S_i$ approaches the boundary of $S_i$. In a joint project with C. Woodward, I am studying the behavior of this moduli space of quilts to determine answers to the following questions:

**Question 1:** Is this moduli space well-behaved? That is, is it smooth, and does it have the expected compactification?

**Question 2:** What is the relationship between this moduli space and the more familiar one without boundary degenerations?
Our analysis thus far suggests that, due to the specific type of boundary degenerations, this moduli space is well-behaved and should be diffeomorphic (at least in the low-dimensional strata) to the usual moduli space. If this proves to be the case, then this boundary-degenerate moduli space can be used to define the symplectic invariants. The upshot would be the following: Recall in the case where $Z = S \times \Sigma$ is a product, the map $\mathcal{N}$ provides a very clear conceptual understanding of the relationship between the two moduli spaces relevant to the Atiyah-Floer conjecture. For more general 4-manifolds, non-canonical choices are introduced to make sense of the term 'holomorphic representative', and the consequence is that we do not have a good map to take the place of $\mathcal{N}$, but only approximations of it. Answers to Questions 1 and 2 above would likely give us a clear canonical notion for 'holomorphic representative', and this would provide a well-defined version of the map $\mathcal{N}$ playing the same role as in the product case.

3 General structure groups

When the Donaldson invariants were first defined for arbitrary smooth 4-manifolds, researchers typically restricted to $G = SU(2)$ or $SO(3)$, together with a specific choice of bundle to avoid reducible connections; see Donaldson [6] and Remark 1(b). However, as more technology was developed, it became clear that one can equally well work with $PU(n)$ as in Kronheimer [20], provided one chooses the bundle appropriately, as before. Similar phenomena appeared in the realm of 3-manifolds. The relevant invariants were first introduced by Floer in [15] and [16] using $G = SU(2)$ and $SO(3)$. Later, extensions to $PU(n)$ were made by Wehrheim and Woodward [36] for symplectic 3-manifold invariants and me [10] for instanton 3-manifold invariants. In fact, the discussion in Sections 1 and 2 has fairly straightforward extensions to $G = PU(n)$; see [10], [12].

Unfortunately, these extensions to the higher rank $PU(n)$ are still not very satisfactory because they continue to place severe restrictions on the choice of principal bundle one can use. That is, the bundle is chosen specifically to avoid the appearance of reducible connections in the analysis, but there are various reasons why this is not always optimal, or even possible. Perhaps the most basic example of this is in the original Atiyah-Floer conjecture described at the end of Section 1 in which one is forced to work with the trivial $SU(2)$ bundle since $Y$ has trivial first homology.

A first step toward rectifying this situation is to carry out many of the symplecto-geometric constructions with orbifolds, and there have been several researchers making strides in this direction; for example, see [3]. However, the relevant spaces have singularities that are worse than orbifold singularities. A long-term research goal of mine is to find a uniform way of dealing with these. One particularly promising approach is to follow a strategy suggested by Chris Woodward in which one carries over the notion of stabilizing divisors [5], [2] to the gauge theoretic world, and combines this with the technique of Manolescu and Woodward [24].

It is conceivable that, if one can get past these technical obstructions, more general structure groups could give rise to new and interesting invariants. However, it is also conceivable that there is nothing new or interesting at all beyond this small collection; see [39]. Regardless, a theorem in either direction would be useful.

In another direction, I would like to carry over various Chern-Simons computations to more general structure groups. The Chern-Simons function is a circle-valued function on the space of connections on a principal $G$-bundle over a 3-manifold $Y$. The critical points of the Chern-Simons function are precisely the flat connections, and the critical values are invariants of the manifold $Y$. In my work on the Atiyah-Floer conjecture I have needed to compute these critical values in the case where $Y$ is the double of a compression body. Since the topology of such a space is fairly simple, I have been able to give a description of the set of components of the space of flat connections. This description has made it possible for me to compute all Chern-Simons critical values [8]. In particular, the techniques I have used apply to arbitrary compact, connected Lie groups $G$.

It is conceivable that these techniques can be extended to more general 3-manifolds as follows: It is well-known that every closed, oriented 3-manifold $Y$ can be obtained from a compression body $C$ by gluing it to itself using a diffeomorphism of the boundary $\phi : \partial C \to \partial C$; write $Y_\phi := Y$ to express this dependence. When $\phi$ is the identity, one obtains a double as addressed in the previous paragraph. Moreover, any bound-

---

1It should be mentioned that a resolution of any of the Atiyah-Floer conjectures via the program described in Section 1 is not dependent on a satisfactory completion of the project described here.
any diffeomorphism $\phi$ is isotopic to a composition $\tau_1 \circ \ldots \circ \tau_k$ of Dehn twists $\tau_i$ in the surface $\partial C$. The idea of the project is to look for a relationship between the Chern-Simons critical values of $Y_{\tau_1 \circ \ldots \circ \tau_k}$ and those of $Y_{(\tau_1 \circ \ldots \circ \tau_k)^{-1}}$. One then hopes to work inductively on $k$ to compare the Chern-Simons critical values of $Y$ with those of the double $D$ (this is $k = 0$). Of course, the difficulty is relating the Chern-Simons critical values of $Y_\phi$ with $Y_{\phi^\tau}$, where $\phi$ is a diffeomorphism of $\partial C$ and $\tau$ is a Dehn twist. However, gauge theory and symplectic geometry have both seen success in this direction. Primary examples being work by Seidel [28] and a generalization by Wehrheim and Woodward [35] that relate the symplectic invariants of $Y_\phi$ with those of $Y_{\phi^\tau}$ in an exact triangle. One long term goal in this direction is to prove an old conjecture that the Chern-Simons critical values are all rational.

4 The quantum cup product in Heegaard Floer theory

In this section, I will describe a joint project with Matt Hedden and John Baldwin that pertains to the Heegaard Floer invariants of a 3-manifold $Y$. To define this invariant, one picks a Heegaard splitting of $Y$. In particular, this identifies an embedded genus $g$ surface $\Sigma \subset Y$, called the Heegaard surface. Then the Heegaard Floer invariant of $Y$ is an abelian group $HF(Y)$ that is defined similar to the symplectic invariants alluded to above, but with the manifold $M(\Sigma)$ replaced by the symmetric product $\text{Sym}^g(\Sigma)$. It turns out that $HF(Y)$ is independent of the choice of Heegaard splitting. One of the main selling features of the Heegaard Floer invariants, as opposed to the Donaldson/symplectic invariants, is that they can often be computed explicitly. This is because the defining PDE tends to reduce to a purely combinatorial problem.

It follows from a general construction that the group $HF(Y)$ is acted upon by the cohomology group $H^*(\text{Sym}^g(\Sigma))$, and this action can be viewed as yet another invariant for $Y$. I will refer to this as the quantum cup product. Matt Hedden asked the following question:

**Question 3:** Can we compute the quantum cup product in any reasonable sense?

It turns out the answer is yes, in the sense that computing the quantum cup product can be carried out in much the same way as traditional computations in Heegaard Floer theory, via Lipshitz’s cylindrical reformulation [22]. To see this, we mimic the work of Piunikhin, Salamon, and Schwarz [25], which provides a very geometric description of this action by first interpreting the cohomology group $H^*(\text{Sym}^g(\Sigma))$ as the Morse homology of a Morse function $\text{Sym}^g(\Sigma) \to \mathbb{R}$. However, the computational power of Heegaard Floer theory comes in reducing questions on the high-dimensional symmetric product $\text{Sym}^g(\Sigma)$ to questions on the surface $\Sigma$. Therefore, we want to relate the Morse theory on the symmetric product with the Morse theory on the surface. To do this, we fix a Morse function $f : \Sigma \to \mathbb{R}$. This induces a function $\text{Sym}^g f : \text{Sym}^g(\Sigma) \to \mathbb{R}$ in a natural way. Unfortunately, the function $\text{Sym}^g f$ is not smooth, so it was not initially clear how to proceed in computing its Morse homology.

The resolution came in viewing $\text{Sym}^g(\Sigma)$ as an orbifold, not as a manifold. Then $\text{Sym}^g f$ is smooth in an orbifold sense. Fortunately, Cho and Hong [4] have recently developed a Morse theory for orbifolds. Their theory is designed as a finite-dimensional model for an infinite-dimensional example in which they are primarily interested. In particular, they make certain very strong non-generic assumptions on the input data. Quite fortunately, our symmetric product set-up satisfies these assumptions spectacularly (this would come as no surprise if the symmetric product was their motivating example, but it seems clear from their paper that it was not). Using their techniques, we can then define the quantum action à la Piunikhin-Salamon-Schwarz, using the function $\text{Sym}^g f$. The upshot is that we can then compute this action by studying the function $f$ on the surface $\Sigma$, which is the affirmative answer to Question 3.

Though $HF(Y)$ is independent of the choice of Heegaard splitting, and hence of the surface $\Sigma \subset Y$, this quantum cup product appears to remember this data. This suggests the quantum cup product could be used to detect various properties of Heegaard surfaces. For example, one could ask:

**Question 4:** Can we use the quantum cup product to obstruct destabilization\(^3\) of the Heegaard splitting?

We are currently investigating questions such as this.

\(^3\)Stabilization of a Heegaard splitting is a procedure that increases the genus of the Heegaard surface $\Sigma$ without changing the ambient manifold $Y$; destabilization is the reverse process.
References


