AXIOM 8.1. For all \( x, y, z \in \mathbb{R} \),
(i) \( x + y = y + x \)
(ii) \( (x + y) + z = x + (y + z) \)
(iii) \( x \cdot (y + z) = x \cdot y + x \cdot z \)
(iv) \( x \cdot y = y \cdot x \)
(v) \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \)

AXIOM 8.2. There exists a number 0, such that for all \( x \in \mathbb{R}, x + 0 = x \)

AXIOM 8.3. There exists a number 1 such that \( 1 \neq 0 \) and whenever \( x \in \mathbb{R}, x \cdot 1 = x \)

AXIOM 8.4. For each \( x \in \mathbb{R} \), there exists a real number, denoted by \( -x \), such that \( x + (-x) = 0 \)

AXIOM 8.5. For each \( x \in \mathbb{R} \setminus \{0\} \), there exists a real number, denoted by \( x^{-1} \), such that \( x \cdot x^{-1} = 1 \)

Define subtraction in \( \mathbb{R} \) by \( x - y = x + (-y) \)

AXIOM 8.26. There exists a subset \( \mathbb{R}^> \) of \( \mathbb{R} \) satisfying
(i) If \( x, y \in \mathbb{R}^> \) then \( x + y \in \mathbb{R}^> \)
(ii) If \( x, y \in \mathbb{R}^> \) then \( x \cdot y \in \mathbb{R}^> \)
(iii) \( 0 \notin \mathbb{R}^> \)
(iv) For every \( x \in \mathbb{R} \), we have \( x \in \mathbb{R}^> \) or \( x = 0 \) or \( -x \in \mathbb{R}^> \)

Members of \( \mathbb{R}^> \) are called positive real numbers. A negative real number is a real number that is neither positive nor zero.

We write \( x < y \) if \( y - x \in \mathbb{R}^> \), and say \( x \) is less than \( y \). Similarly we write \( x \leq y \) if \( y - x \in \mathbb{R}^> \) or \( x = y \), and say \( x \) is less than or equal to \( y \).

Let \( A \) be a nonempty subset of \( \mathbb{R} \). The set \( A \) is bounded above if there exists \( b \in \mathbb{R} \) such that for all \( a \in A \), \( a \leq b \). Any such number \( b \) is called an upper bound for \( A \). If \( b \) is an upper bound for \( A \) that is less than any other upper bound for \( A \), it is called a least upper bound for \( A \) and is denoted by sup(\( A \)) (sup is an abbreviation for supremum).

Note that so far \( \mathbb{Q} \) satisfies all the axioms we have listed. For subsets of \( \mathbb{Q} \), supremum might not exist within rational numbers such as for \( A = \{ x \in \mathbb{Q} \mid x^2 < 3 \} \). To characterize real numbers we require one more axiom to be satisfied:

AXIOM 8.52. (Completeness axiom). Every nonempty subset of \( \mathbb{R} \) that is bounded above has a least upper bound.

So far we have notations for only two special real numbers: 0 and 1. Next we define \( 2 = 1 + 1 \), \( 3 = 2 + 1 \), ..., \( 9 = 8 + 1 \), which are called digits. Natural numbers within the set of real numbers is defined by finite sums of the form \( 1 + 1 + ... + 1 \), in particular the natural number \( n \) corresponds to the sum of \( n \) copies of 1.

THEOREM 8.42. \( \mathbb{R}^> \) does not have a smallest element.

THEOREM 10.1. The set of natural numbers as a subset of \( \mathbb{R} \) is not bounded above in \( \mathbb{R} \).

PROPOSITION 10.4. For each \( \epsilon \in \mathbb{R}^> \), there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \epsilon \).

PROPOSITION 10.11. Let \( x, y \in \mathbb{R} \). Then \( x = y \) if and only if for every \( \epsilon > 0 \) we have \( |x - y| \leq \epsilon \).

EXERCISE 1. Using these axioms, show that \((x \cdot y)^{-1} = x^{-1} \cdot y^{-1} \).

EXERCISE 2. Using these axioms, show that 1 is a positive real number. (Hint: use proof by contradiction, Axiom 8.26(iv) and Axiom 8.26(ii))