1. Use induction to prove that

\[ 1 + 3 + 6 + \cdots + \frac{n(n + 1)}{2} = \frac{n(n + 1)(n + 2)}{6} \]

for all \( n \in \mathbb{N} \).

2. Use induction to prove that \( 7 \mid (9^n - 2^n) \) for every \( n \in \mathbb{N} \).

3. Use the Strong Principle of Mathematical Induction to prove that for each integer \( n \geq 13 \), there are nonnegative integers \( x \) and \( y \) such that \( n = 3x + 4y \).

4. Define a sequence \( a_n \) by \( a_1 = 3, a_2 = 5 \), and

\[ a_n = 3a_{n-1} - 2a_{n-2} \text{ for } n \geq 3. \]

Show \( a_n = 2^n + 1 \) for all \( n \in \mathbb{N} \).

5. Let \( R \) be a relation defined on \( \mathbb{N}^2 \) by \( (a, b)R(c, d) \) if \( ad = bc \). Prove or disprove that \( R \) is an equivalence relation. If it is an equivalence relation, describe the elements in the equivalence class \([ (1, 2) ]\).

6. For \( (a, b) \) and \( (c, d) \in \mathbb{R}^2 \), define \( (a, b) \sim (c, d) \) if \( |a| = |c| \) and \( |b| = |d| \), where \( |x| \) is the greatest integer less than or equal to \( x \). Prove or disprove that a relation \( \sim \) is an equivalence relation in \( \mathbb{R}^2 \).

7. A relation \( R \) is defined on the set of positive rational numbers by \( aRb \) if \( \frac{a}{b} \in \{3^k : k \in \mathbb{Z}\} \). Prove that a relation \( R \) is an equivalence relation and describe the elements in the equivalence class of 2.

8. (a) Fill in the following addition and multiplication tables for \( \mathbb{Z}_4 \).

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(b) For each of the following modular arithmetic equations, use the tables above to either find all solutions or explain why it has no solution. The coefficients and variables should be taken in \( \mathbb{Z}_4 \).


9. Let \([a], [b] \in \mathbb{Z}_{11}\) and \([a] \neq [0]\). Prove that the equation \([a]x + [b] = 0\) always has exactly one solution.

10. Let \([a], [b] \in \mathbb{Z}_{11}\), and assume \([a] \cap [b] \neq \emptyset\). Prove that \([a] = [b]\).

11. Fill in the blanks in part (a).

(a) Let \(A\) and \(B\) be sets. A relation \(R \subset A \times B\) defines a function from \(A\) to \(B\) if

(1) \( \forall a \in A, \exists b \in B\) such that \(\ldots\) and

(2) \( \forall a \in A, \forall b_1, b_2 \in B, \) if \((a, b_1) \in R\) and \((a, b_2) \in R\), then \(\ldots\)

(b) Let \(A\) be a set. Prove that there exists a unique relation \(R\) on \(A\) such that \(R\) is an equivalence relation on \(A\) and \(R\) is a function from \(A\) to \(A\).

(c) Let \(n \in \mathbb{N}\) and \(a, b \in \mathbb{Z}\). Prove that the following relation \(R\) defines a function from \(\mathbb{Z}_n\) to \(\mathbb{Z}_n\).

\[ R = \{([x], [ax + b]) \mid x \in \mathbb{Z}\}. \]

12. Let \(A\) and \(B\) be sets. Suppose that \(f : A \to B\) is a function. Let \(C, D \subseteq B\). Prove that

\[ f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D). \]

13. Let \(\mathbb{Z}_7 = \{[0], [1], \ldots, [6]\}\) be the set of congruence classes of integers modulo 7 together with the operations of addition and multiplication of congruence classes. Suppose that \(f : \mathbb{Z}_7 \to \mathbb{Z}_7\) is the function defined by the rule

\[ f([x]) = [2x + 1] \text{ for each } x \in \mathbb{Z}. \]

(a) Prove that \(f\) is a bijection.

(b) Prove that there exists integers \(a\) and \(b\) such that \(f^{-1}\) is given by the rule

\[ f^{-1}([x]) = [ax + b] \text{ for each } x \in \mathbb{Z}. \]

14. Find an appropriate domain \(A \subseteq \mathbb{R}\) and codomain \(B \subseteq \mathbb{R}\) so that \(f : A \to B\) defined by \(f(x) = x^2 - 4x + 1\) is a bijection.