1. Finish the proof from class of the monotone convergence theorem by showing that if \((x_n)\) is a sequence that is bounded below and decreasing, then \((x_n)\) converges.

Proof: Assume \((x_n)\) is bounded below and decreasing. Since it is bounded below, the infimum \(L := \inf \{x_n\}\) exists. We will show \(\lim x_n = L\). Let \(\epsilon > 0\). Then \(L + \epsilon > L\) and so \(L + \epsilon\) cannot be a lower bound for \(\{x_n\}\). It follows that there is some \(N\) such that \(x_N < L + \epsilon\). Since \((x_n)\) is decreasing, if \(n \geq N\) then \(x_n \leq x_N\), and so \(x_n < L + \epsilon\). Rearranging this, and using the fact that \(L \leq x_n\) for all \(n\), gives

\[|x_n - L| = x_n - L < \epsilon\]

whenever \(n \geq N\). Q.E.D.

2. Show that if \(\lim_{n \to \infty} x_n = \infty\), then \((x_n)\) diverges. Recall that, by definition, \(\lim_{n \to \infty} x_n = \infty\) if for all \(B \in \mathbb{R}\), there is some \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(x_n \geq B\).

Assume \(\lim_{n \to \infty} x_n = \infty\), and let \(L \in \mathbb{R}\). It suffices to show that \((x_n)\) does not converge to \(L\). Take \(\epsilon = 1\), and let \(N\) be any natural number. Using the definition of \(\lim_{n \to \infty} x_n = \infty\), with \(B = L\), there is some \(N'\) such that \(x_n \geq L + 1\) whenever \(n \geq N'\). Then let \(n\) be any integer greater than \(N\) and \(N'\). It follows that

\[|x_n - L| = x_n - L \geq 1 = \epsilon\]

This proves that \((x_n)\) does not converge to \(L\), as desired. Q.E.D.

3. Suppose that the series \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) both converge. Show that \(\sum_{n=1}^{\infty} (a_n + b_n)\) converges and its value equals \(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\).

Proof: Assume that \(\sum_{n=1}^{\infty} a_n\) and \(\sum_{n=1}^{\infty} b_n\) both converge, and let \(A, B\) be the values that they converge to. Let \(s_k := \sum_{n=1}^{k} a_n\) and \(t_k := \sum_{n=1}^{k} b_n\) denote the partial sums. The assumption that the series converge to \(A\) and \(B\), respectively, is equivalent to saying that the sequences \((s_k)_k\) and \((t_k)_k\) converge to \(A\) and \(B\), respectively. Then by Beck Proposition 10.23 (iii), we have

\[\lim_k s_k + t_k = \lim_k s_k + \lim_k t_k = A + B\]

This is exactly the statement that \(\sum_n a_n + b_n\) converges to \(A + B\), since \(s_k + t_k\) is the \(k\)th partial sum of \(\sum_n a_n + b_n\). Q.E.D.