From instantons to quilts with seam degenerations

David L. Duncan

Michigan State University

2014 CMS Winter Meeting
Notation

$\Sigma$; closed, connected, oriented surface with Riemannian metric

$P \to \Sigma$; principal $SO(3)$-bundle

Assume $P$ is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with $PU(r)$-bundles, $r \geq 2$, provided the degree is coprime to $r$.)
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

Notation

$\Sigma$; closed, connected, oriented surface with Riemannian metric

$P \to \Sigma$; principal $SO(3)$-bundle

Assume $P$ is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with $PU(r)$-bundles, $r \geq 2$, provided the degree is coprime to $r$.)
Notation

\[ \Sigma; \] closed, connected, oriented surface with Riemannian metric

\[ P \to \Sigma; \] principal SO(3)-bundle

Assume \( P \) is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with \( PU(r) \)-bundles, \( r \geq 2 \), provided the degree is coprime to \( r \).)
\[ \Sigma; \text{closed, connected, oriented surface with Riemannian metric} \]

\[ P \to \Sigma; \text{principal } \text{SO}(3)-\text{bundle} \]

Assume \( P \) is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with \( \text{PU}(r) \)-bundles, \( r \geq 2 \), provided the degree is coprime to \( r \).)
\[\Sigma; \text{closed, connected, oriented surface with Riemannian metric}\]

\[P \rightarrow \Sigma; \text{principal SO}(3)\text{-bundle}\]

Assume \(P\) is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with \(\text{PU}(r)\)-bundles, \(r \geq 2\), provided the degree is coprime to \(r\).)
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
- Local picture
- Global picture

More general 4-manifolds
- Morse
- 2-functions
- Quilts
- Quilts from instantons

History

Theorem (Dostoglou-Salamon (1993))

$HF_{inst}(S^1 \times \Sigma)$ defined by counting (isolated) instantons on $R \times S^1 \times \Sigma$.

$HF_{symp}(S^1 \times \Sigma)$ defined by counting (isolated) holomorphic curves $R \times S^1 \rightarrow M(\Sigma)$, where $M(\Sigma) = \{\text{flat connections}\}$. gauges
\[ HF_{\text{inst}}(S^1 \times \Sigma) \]

defined by counting (isolated) instantons on \( \mathbb{R} \times S^1 \times \Sigma \)
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

\[ HF_{\text{inst}}(S^1 \times \Sigma) \]
defined by counting (isolated) instantons on $\mathbb{R} \times S^1 \times \Sigma$

\[ HF_{\text{symp}}(S^1 \times \Sigma) \]
defined by counting (isolated) holomorphic curves $\mathbb{R} \times S^1 \rightarrow M(\Sigma)$,
History

\[ HF_{\text{inst}}(S^1 \times \Sigma) \]
defined by counting (isolated) instantons on \( \mathbb{R} \times S^1 \times \Sigma \)

\[ HF_{\text{symp}}(S^1 \times \Sigma) \]
defined by counting (isolated) holomorphic curves

\[ \mathbb{R} \times S^1 \rightarrow M(\Sigma), \]

where

\[ M(\Sigma) = \{ \text{flat connections} \} \]

\[ \text{gauge} \]
Theorem (Dostoglou-Salamon (1993))

\[ \text{HF}_{\text{inst}}(S^1 \times \Sigma) \quad \overset{\equiv}{=} \quad \text{HF}_{\text{symp}}(S^1 \times \Sigma) \]

defined by counting (isolated) instantons on \( \mathbb{R} \times S^1 \times \Sigma \)
defined by counting (isolated) holomorphic curves

\[ \mathbb{R} \times S^1 \rightarrow M(\Sigma), \]

where

\[ M(\Sigma) = \left\{ \text{flat connections} \right\} / \text{gauge} \]
The proof reduces to defining a bijection

\[ \{ \text{(isolated) holo. maps } \mathbb{R} \times S^1 \to M(\Sigma) \} \]
\[ \longrightarrow \{ \text{(isolated) instantons on } \mathbb{R} \times S^1 \times \Sigma \} / \text{gauge} \]

for suitable auxiliary data (metric, perturbation).

Dostoglou-Salamon define this map using an implicit function theorem.
The proof reduces to defining a bijection

$$\{(\text{isolated}) \text{ holo. maps } \mathbb{R} \times S^1 \to M(\Sigma)\} \longrightarrow \{(\text{isolated}) \text{ instantons on } \mathbb{R} \times S^1 \times \Sigma\} / \text{gauge}$$

for suitable auxiliary data (metric, perturbation).

Dostoglou-Salamon define this map using an implicit function theorem.
History

The proof reduces to defining a bijection

\[
\{(\text{isolated}) \text{ holo. maps } \mathbb{R} \times S^1 \to M(\Sigma)\} 
\longrightarrow \{(\text{isolated}) \text{ instantons on } \mathbb{R} \times S^1 \times \Sigma\} / \text{gauge}
\]

for suitable auxiliary data (metric, perturbation).

Dostoglou-Salamon define this map using an implicit function theorem.
Their theorem continues to hold if $\mathbb{R} \times S^1 \times \Sigma$ is replaced by $\mathbb{R} \times (\text{mapping torus})$.

The perspective of this talk is to view their theorem as a statement about 4-manifolds.

Question: Can we extend this to more general manifolds?
Their theorem continues to hold if $\mathbb{R} \times S^1 \times \Sigma$ is replaced by $\mathbb{R} \times (\text{mapping torus})$.

The perspective of this talk is to view their theorem as a statement about 4-manifolds.

Question: Can we extend this to more general manifolds?
Their theorem continues to hold if $\mathbb{R} \times S^1 \times \Sigma$ is replaced by $\mathbb{R} \times (\text{mapping torus})$.

The perspective of this talk is to view their theorem as a statement about 4-manifolds.

Question: Can we extend this to more general manifolds?
Summary of results

Replace $\mathbb{R} \times S^1 \times \Sigma$ with an oriented, connected 4-manifold $Z$ with cylindrical ends (allowing the possibility of no ends).

Replace $\mathbb{R} \times S^1$ with an oriented surface $S$ having the same number of cylindrical ends.

Assume there is a map $F : Z \to S$ that preserves the ends and has connected, non-empty fibers.

Assume there is an $SO(3)$-bundle $R \to Z$ that restricts to the non-trivial bundle on some fiber of $F$. 
Summary of results

Replace $\mathbb{R} \times S^1 \times \Sigma$ with an oriented, connected 4-manifold $Z$ with cylindrical ends (allowing the possibility of no ends).

Replace $\mathbb{R} \times S^1$ with an oriented surface $S$ having the same number of cylindrical ends.

Assume there is a map $F : Z \rightarrow S$ that preserves the ends and has connected, non-empty fibers.

Assume there is an $SO(3)$-bundle $R \rightarrow Z$ that restricts to the non-trivial bundle on some fiber of $F$. 
Summary of results

Replace \( \mathbb{R} \times S^1 \times \Sigma \) with an oriented, connected 4-manifold \( Z \) with cylindrical ends (allowing the possibility of no ends).

Replace \( \mathbb{R} \times S^1 \) with an oriented surface \( S \) having the same number of cylindrical ends.

Assume there is a map \( F : Z \to S \) that preserves the ends and has connected, non-empty fibers.

Assume there is an \( SO(3) \)-bundle \( R \to Z \) that restricts to the non-trivial bundle on some fiber of \( F \).
Summary of results

Replace $\mathbb{R} \times S^1 \times \Sigma$ with an oriented, connected 4-manifold $Z$ with cylindrical ends (allowing the possibility of no ends).

Replace $\mathbb{R} \times S^1$ with an oriented surface $S$ having the same number of cylindrical ends.

Assume there is a map $F : Z \to S$ that preserves the ends and has connected, non-empty fibers.

Assume there is an $SO(3)$-bundle $R \to Z$ that restricts to the non-trivial bundle on some fiber of $F$. 
Summary of results

Assume for now that $F$ is a fiber bundle.
The more general case will be discussed briefly at the end.

For $x \in S$, let $\Sigma_x = F^{-1}(x)$ be the fiber over $x$.

Let $M(Z)$ be the fiber bundle over $S$ with fiber over $x \in S$
given by $M(\Sigma_x)$. 
Assume for now that $F$ is a fiber bundle.

The more general case will be discussed briefly at the end.

For $x \in S$, let $\Sigma_x = F^{-1}(x)$ be the fiber over $x$.

Let $M(Z)$ be the fiber bundle over $S$ with fiber over $x \in S$ given by $M(\Sigma_x)$. 
Assume for now that $F$ is a fiber bundle.

The more general case will be discussed briefly at the end.

For $x \in S$, let $\Sigma_x = F^{-1}(x)$ be the fiber over $x$.

Let $M(Z)$ be the fiber bundle over $S$ with fiber over $x \in S$ given by $M(\Sigma_x)$. 
Summary of results

Theorem (D. (2014))

There is an injection

\[ M : \{ \text{(isolated) holo. sections } S \to M(Z) \} \rightarrow \{ \text{(isolated) instantons on } Z \} / \text{gauge} \]

for suitable auxiliary data (metric, perturbation).
Summary of results

Theorem (D. (2014))

*There is an injection*

\[ \mathcal{M} : \{(isolated) \ holo. \ sections \ S \rightarrow M(Z)\} \rightarrow \{(isolated) \ instantons \ on \ Z\} / gauge \]

*for suitable auxiliary data (metric, perturbation).*

Either use the Dostoglou-Salamon implicit function theorem, or use the Yang-Mills heat flow. (The latter generalizes to when \( F \) is not a fiber bundle.)
Summary of results

**Theorem (D. (2013))**

There is a map

\[
\{(\text{isolated) holo. sections } S \to M(Z)\} \\
\leftarrow \{(\text{isolated) instantons on } Z\} / \text{gauge : } N
\]

that is well-defined for the same auxiliary data (metric, perturbation) as $M$. 

Summary of results

Theorem (D. (2013))

There is a map

\[
\{(\text{isolated) holo. sections } S \rightarrow M(Z)\} \\
\leftarrow \{(\text{isolated) instantons on } Z\} / \text{gauge : } N
\]

that is well-defined for the same auxiliary data (metric, perturbation) as \( M \).

Considered independently by Nishinou in the case where \( Z \) is a product of surfaces [N].
Summary of results

Theorem (D. (2013))

There is a map

\[
\{(isolated)\ holo.\ sections\ S \rightarrow M(Z)\} \\
\leftarrow \{(isolated)\ instantons\ on\ Z\} / gauge : N
\]

that is well-defined for the same auxiliary data (metric, perturbation) as \(M\).

Considered independently by Nishinou in the case where \(Z\) is a product of surfaces [N].

Theorem (D.-McNamara (2014))

The map \(N\) is an injection.
Summary of results

Corollary

The relative Donaldson invariant of \((Z, R, F)\) agrees with the relative symplectic invariant of \((Z, R, F)\).

Generalizes Dostoglou-Salamon’s theorem, and gives an alternate proof.

‘Relative’ refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when \(Z\) is closed.

Corollary

The relative symplectic invariant of \((Z, R, F)\) depends only on \(F\) through its homotopy class.
Summary of results

Corollary

The relative Donaldson invariant of \((Z, R, F)\) agrees with the relative symplectic invariant of \((Z, R, F)\).

Generalizes Dostoglou-Salamon’s theorem, and gives an alternate proof.

‘Relative’ refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when \(Z\) is closed.

Corollary

The relative symplectic invariant of \((Z, R, F)\) depends only on \(F\) through its homotopy class.
Summary of results

Corollary

The relative Donaldson invariant of \((Z, R, F)\) agrees with the relative symplectic invariant of \((Z, R, F)\).

Generalizes Dostoglou-Salamon’s theorem, and gives an alternate proof.

‘Relative’ refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when \(Z\) is closed.

Corollary

The relative symplectic invariant of \((Z, R, F)\) depends only on \(F\) through its homotopy class.
Summary of results

Corollary

The relative Donaldson invariant of \((Z, R, F)\) agrees with the relative symplectic invariant of \((Z, R, F)\).

Generalizes Dostoglou-Salamon’s theorem, and gives an alternate proof.

‘Relative’ refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when \(Z\) is closed.

Corollary

The relative symplectic invariant of \((Z, R, F)\) depends only on \(F\) through its homotopy class.
Summary of results

Corollary

The relative Donaldson invariant of \((Z, R, F)\) agrees with the relative symplectic invariant of \((Z, R, F)\).

Generalizes Dostoglou-Salamon’s theorem, and gives an alternate proof.

‘Relative’ refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when \(Z\) is closed.

Corollary

The relative symplectic invariant of \((Z, R, F)\) depends only on \(F\) through its homotopy class.
Outline of the rest of the talk
Outline of the rest of the talk

- Define $\mathcal{N}$ for fiber bundles.

This uses the Narasimhan-Seshadri correspondence for surfaces.
Outline of the rest of the talk

- Define $\mathcal{N}$ for fiber bundles. This uses the Narasimhan-Seshadri correspondence for surfaces.

- Discuss non-fiber bundle case. This is where quilts enter the picture (e.g., the *quilted Atiyah-Floer conjecture*, and its 4-manifold generalizations).
Connections

\[ P \to \Sigma \text{ as above} \]
\[ \mathcal{A}(\Sigma) = \{ \text{connections on } P \} \]
\[ \mathcal{A}_{\text{flat}}(\Sigma) = \{ \alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0 \} \]
where \( F_\alpha \) is the curvature of \( \alpha \)
\[ \mathcal{G}(\Sigma) = \{ \text{gauge transformations on } P \} \]
\[ \mathcal{G}_0(\Sigma) = \{ g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id} \} \]
The action of \( \mathcal{G}_0(\Sigma) \) on \( \mathcal{A}_{\text{flat}}(\Sigma) \) is free, and
\[ \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma) \]
is a smooth finite-dimensional symplectic manifold.
Connections

\[ P \to \Sigma \text{ as above} \]

\[ \mathcal{A}(\Sigma) = \{ \text{connections on } P \} \]

\[ \mathcal{A}_{\text{flat}}(\Sigma) = \{ \alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0 \} \]

where \( F_\alpha \) is the curvature of \( \alpha \)

\[ \mathcal{G}(\Sigma) = \{ \text{gauge transformations on } P \} \]

\[ \mathcal{G}_0(\Sigma) = \{ g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id} \} \]

The action of \( \mathcal{G}_0(\Sigma) \) on \( \mathcal{A}_{\text{flat}}(\Sigma) \) is free, and

\[ \mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma) \]

is a smooth finite-dimensional symplectic manifold.
From instantons to quilts with seam degenerations

David L. Duncan

Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}$

where $F_\alpha$ is the curvature of $\alpha$

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$

is a smooth finite-dimensional symplectic manifold.
From instantons to quilts with seam degenerations

David L. Duncan

Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}
\quad \text{where } F_\alpha \text{ is the curvature of } \alpha$

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$

is a smooth finite-dimensional symplectic manifold.
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}$

where $F_\alpha$ is the curvature of $\alpha$

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$

is a smooth finite-dimensional symplectic manifold.
Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}$

where $F_\alpha$ is the curvature of $\alpha$

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$

is a smooth finite-dimensional symplectic manifold.
Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}$

where $F_\alpha$ is the curvature of $\alpha$

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

$\mathcal{A}_{\text{flat}}(\Sigma)/\mathcal{G}_0(\Sigma)$

is a smooth finite-dimensional symplectic manifold.
Complexified gauge group

Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \to \Sigma\}$

$G^C$ complexification of $G = \text{SO}(3)$

$G^C = \{\text{sections of } P \times_G G^C \to \Sigma\}$

$G^C_0 := \{g \in G^C \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $G^C_0$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 


Complexified gauge group

Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \to \Sigma\}$

$G^C$ complexification of $G = \text{SO}(3)$

$
\mathcal{G}^C = \{\text{sections of } P \times_G G^C \to \Sigma\}
$

$\mathcal{G}_0^C := \{g \in \mathcal{G}^C \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $\mathcal{G}_0^C$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 
Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \to \Sigma\}$

$G^C$ complexification of $G = \text{SO}(3)$

$G^C = \{\text{sections of } P \times_G G^C \to \Sigma\}$

$G^C_0 := \{g \in G^C \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $G^C_0$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 

Complexified gauge group
Complexified gauge group

Recall \( \mathcal{G}(\Sigma) = \{ \text{sections of } P \times_G G \to \Sigma \} \)

\( G^\mathbb{C} \) complexification of \( G = \text{SO}(3) \)

\[ G^\mathbb{C} = \{ \text{sections of } P \times_G G^\mathbb{C} \to \Sigma \} \]

\[ G^\mathbb{C}_0 := \{ g \in G^\mathbb{C} | g \simeq \text{Id} \} \]

These act on \( \mathcal{A}(\Sigma) \) by identifying \( \mathcal{A}(\Sigma) \) with the space of Cauchy-Riemann operators.

There is a preferred neighborhood \( \mathcal{A}^{ss}(\Sigma) \) of \( \mathcal{A}_{\text{flat}}(\Sigma) \) on which \( G^\mathbb{C}_0 \) acts freely.

There is some \( \epsilon_0 > 0 \) so that if \( \| F_\alpha \|_{L^2} \leq \epsilon_0 \), then \( \alpha \in \mathcal{A}^{ss}(\Sigma) \).
Complexified gauge group

Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \to \Sigma\}$

$G^\mathbb{C}$ complexification of $G = \text{SO}(3)$

$\mathcal{G}^\mathbb{C} = \{\text{sections of } P \times_G G^\mathbb{C} \to \Sigma\}$

$\mathcal{G}_0^\mathbb{C} := \{g \in \mathcal{G}^\mathbb{C} \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $\mathcal{G}_0^\mathbb{C}$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 
Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \to \Sigma\}$

$G^\mathbb{C}$ complexification of $G = \text{SO}(3)$

$\mathcal{G}^\mathbb{C} = \{\text{sections of } P \times_G G^\mathbb{C} \to \Sigma\}$

$\mathcal{G}_0^\mathbb{C} := \{g \in \mathcal{G}^\mathbb{C} | g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $\mathcal{G}_0^\mathbb{C}$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 
Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_{\mathbb{G}} \mathbb{G} \to \Sigma\}$

$\mathbb{G}^C$ complexification of $\mathbb{G} = \text{SO}(3)$

$\mathcal{G}^C = \{\text{sections of } P \times_{\mathbb{G}} \mathbb{G}^C \to \Sigma\}$

$\mathcal{G}^C_0 := \{g \in \mathcal{G}^C \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $\mathcal{G}^C_0$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 
Complexified gauge group

Recall $\mathcal{G}(\Sigma) = \{\text{sections of } P \times_G G \rightarrow \Sigma\}$

$G^\mathbb{C}$ complexification of $G = \text{SO}(3)$

$\mathcal{G}^\mathbb{C} = \{\text{sections of } P \times_G G^\mathbb{C} \rightarrow \Sigma\}$

$\mathcal{G}_0^\mathbb{C} := \{g \in \mathcal{G}^\mathbb{C} \mid g \simeq \text{Id}\}$

These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\text{flat}}(\Sigma)$ on which $\mathcal{G}_0^\mathbb{C}$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$. 
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary
Def. of $\mathcal{N}$
Local picture
Global picture
More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
- Local picture
- Global picture

More general 4-manifolds
- Morse
- 2-functions
- Quilts
- Quilts from instantons

\[ \frac{A^{ss}(\Sigma)}{G_0^C} \]
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse
2-functions
Quilts
Quilts from instantons

\[
\frac{A_{\text{flat}}(\Sigma)}{\mathcal{G}_0(\Sigma)} \quad \frac{A^{ss}(\Sigma)}{\mathcal{G}_0^C}
\]
Theorem (Narasimhan-Seshadri (1965))

\[ \frac{A_{\text{flat}}(\Sigma)}{G_0(\Sigma)} \cong \frac{A^{ss}(\Sigma)}{G_0^C} \]
Theorem (Narasimhan-Seshadri (1965))

\[ M(\Sigma) := \frac{A_{\text{flat}}(\Sigma)}{G_0(\Sigma)} \supseteq \frac{A^{ss}(\Sigma)}{G_0^C} \]
Theorem (Narasimhan-Seshadri (1965))

\[ M(\Sigma) := \frac{A_{\text{flat}}(\Sigma)}{G_0(\Sigma)} \cong \frac{A^{ss}(\Sigma)}{G_0^C} \]

Denote the projection by

\[ \Pi : A^{ss}(\Sigma) \longrightarrow M(\Sigma) \]
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  $\Rightarrow A(\Sigma)$ has a complex structure given by $*$

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $*$. $\Pi$ preserves this structure:

$$D_\alpha \Pi(*\mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  $\Rightarrow A(\Sigma)$ has a complex structure given by $*$

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $*$. $\Pi$ preserves this structure:

$$D_\alpha \Pi(*\mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $\ast$
- $\ast^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  $\Rightarrow A(\Sigma)$ has a complex structure given by $\ast$

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $\ast$.

$\Pi$ preserves this structure:

$$D_\alpha \Pi(\ast \mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$

$\Rightarrow A(\Sigma)$ has a complex structure given by $*$.

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $*$.

$\Pi$ preserves this structure:

$$D_\alpha \Pi(*\mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $\ast$
- $\ast^2 = -1$ on 1-forms
- $\mathcal{T}_\alpha \mathcal{A}(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  \[ \Rightarrow \mathcal{A}(\Sigma) \text{ has a complex structure given by } \ast \]

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $\ast$.

$\Pi$ preserves this structure:

\[ D_\alpha \Pi(\ast \mu) = J D_\alpha \Pi(\mu) \]

where $D_\alpha \Pi$ is the linearization at $\alpha \in \mathcal{A}^{ss}(\Sigma)$ and $\mu \in T_\alpha \mathcal{A}(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  $\Rightarrow A(\Sigma)$ has a complex structure given by $*$

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $*$. $\Pi$ preserves this structure:

$$D_\alpha \Pi(*\mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
A property of $\Pi$

- The metric and orientation on $\Sigma$ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha A(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$
  $\Rightarrow A(\Sigma)$ has a complex structure given by $*$

Similarly, $M(\Sigma)$ has a complex structure $J$ induced from $*$. $\Pi$ preserves this structure:

$$D_\alpha \Pi(*\mu) = J D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in A^{ss}(\Sigma)$ and $\mu \in T_\alpha A(\Sigma)$. 
Corollaries

- The map $\Pi$ takes holomorphic curves in $A^s_s(\Sigma)$ to holomorphic curves in $M(\Sigma)$.

- This can be strengthened as follows: The tangent space to the complex gauge orbit through $\alpha$ is

$$\ker D_\alpha \Pi = \text{Im } d_\alpha \oplus \text{Im } \ast d_\alpha,$$

where $d_\alpha$ is the covariant derivative of $\alpha$ in the adjoint representation.
Corollaries

• The map $\Pi$ takes holomorphic curves in $A^{ss}(\Sigma)$ to holomorphic curves in $M(\Sigma)$.

• This can be strengthened as follows:
The tangent space to the complex gauge orbit through $\alpha$ is

$$\ker D_\alpha \Pi = \text{Im } d_\alpha \oplus \text{Im } * d_\alpha,$$

where $d_\alpha$ is the covariant derivative of $\alpha$ in the adjoint representation.
Corollaries

- The map $\Pi$ takes holomorphic curves in $A^{ss}(\Sigma)$ to holomorphic curves in $M(\Sigma)$.

- This can be strengthened as follows:
The tangent space to the complex gauge orbit through $\alpha$ is

$$\ker D_\alpha \Pi = \text{Im } d_\alpha \oplus \text{Im } \ast d_\alpha,$$

where $d_\alpha$ is the covariant derivative of $\alpha$ in the adjoint representation.
Corollaries

- The map $\Pi$ takes holomorphic curves in $\mathcal{A}^{ss}(\Sigma)$ to holomorphic curves in $M(\Sigma)$.
- This can be strengthened as follows:
The tangent space to the complex gauge orbit through $\alpha$ is

$$\ker D_{\alpha} \Pi = \text{Im} \ d_{\alpha} \oplus \text{Im} \ast d_{\alpha},$$

where $d_{\alpha}$ is the covariant derivative of $\alpha$ in the adjoint representation.
Corollaries

- The map $\Pi$ takes holomorphic curves in $A^{ss}(\Sigma)$ to holomorphic curves in $M(\Sigma)$.

- This can be strengthened as follows:
The tangent space to the complex gauge orbit through $\alpha$ is

$$\ker D_\alpha \Pi = \text{Im } d_\alpha \oplus \text{Im } * d_\alpha,$$

where $d_\alpha$ is the covariant derivative of $\alpha$ in the adjoint representation.
In particular, if $\alpha : S \subseteq \mathbb{C} \rightarrow A(\Sigma)$ satisfies
\[
\partial_s \alpha \oplus \ast \partial_t \alpha \in \text{Im } d\alpha \oplus \text{Im } \ast d\alpha,
\]
and
\[
\sup_{S} \| F_\alpha \|_{L^2(\Sigma)} \leq \epsilon_0,
\]
then $\Pi(\alpha) : S \rightarrow M(\Sigma)$ is holomorphic.
In particular, if $\alpha : S \subseteq \mathbb{C} \to \mathcal{A}(\Sigma)$ satisfies
\[ \partial_s \alpha + * \partial_t \alpha \in \text{Im} \, d\alpha \oplus \text{Im} \, * \, d\alpha, \]
and
\[ \sup_S \| F_\alpha \|_{L^2(\Sigma)} \leq \epsilon_0, \]
then $\Pi(\alpha) : S \to M(\Sigma)$ is holomorphic.
Corollaries

In particular, if $\alpha : S \subseteq \mathbb{C} \to \mathcal{A}(\Sigma)$ satisfies

$$\partial_s \alpha + \ast \partial_t \alpha \in \text{Im } d\alpha \oplus \text{Im } \ast d\alpha,$$

and

$$\sup_S \|F_{\alpha}\|_{L^2(\Sigma)} \leq \epsilon_0,$$

then $\Pi(\alpha) : S \to M(\Sigma)$ is holomorphic.
Local picture: The case $S \times \Sigma$

\[\Sigma \text{ as above}\]
\[S \subset \mathbb{C}\]
\[\mathcal{Z} := S \times \Sigma \text{ with induced orientation and metric}\]
Local picture: The case $S \times \Sigma$

$\Sigma$ as above

$S \subset \mathbb{C}$

$Z := S \times \Sigma$ with induced orientation and metric
Local picture: The case $S \times \Sigma$

$\Sigma$ as above

$S \subset \mathbb{C}$

$Z := S \times \Sigma$ with induced orientation and metric
Local picture: The case $S \times \Sigma$

$\Sigma$ as above

$S \subset \mathbb{C}$

$\mathcal{Z} := S \times \Sigma$ with induced orientation and metric
Local picture: The case $S \times \Sigma$

Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection.

$(s, t)$ coordinates on $S \subset \mathbb{C}$.

Write $A$ in coordinates

$$A = \alpha + \phi \, ds + \psi \, dt,$$

$\alpha : S \to \mathcal{A}(\Sigma)$ defined by $\alpha(s, t) := A|_{(s,t) \times \Sigma}$

$\phi, \psi$ defined by contracting $A$ with $\partial_s, \partial_t$, respectively.
Local picture: The case $S \times \Sigma$

Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection. $(s, t)$ coordinates on $S \subset \mathbb{C}$. Write $A$ in coordinates

$$A = \alpha + \phi \, ds + \psi \, dt,$$

$\alpha : S \to \mathcal{A}(\Sigma)$ defined by $\alpha(s, t) := A|_{\{(s, t)\} \times \Sigma}$

$\phi, \psi$ defined by contracting $A$ with $\partial_s, \partial_t$, respectively.
Local picture: The case $S \times \Sigma$

Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection. 

$(s, t)$ coordinates on $S \subset \mathbb{C}$.

Write $A$ in coordinates

$$A = \alpha + \phi \, ds + \psi \, dt,$$

$\alpha : S \to \mathcal{A}(\Sigma)$ defined by $\alpha(s, t) := A|_{\{(s, t)\} \times \Sigma}$

$\phi, \psi$ defined by contracting $A$ with $\partial_s, \partial_t$, respectively.
Local picture: The case $S \times \Sigma$

Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection. (s, t) coordinates on $S \subset \mathbb{C}$.

Write $A$ in coordinates

$$A = \alpha + \phi \, ds + \psi \, dt,$$

$\alpha : S \to \mathcal{A}(\Sigma)$ defined by $\alpha(s, t) := A|\{(s, t)\} \times \Sigma$

$\phi, \psi$ defined by contracting $A$ with $\partial_s, \partial_t$, respectively.
Local picture: The case $S \times \Sigma$

Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection. (s, t) coordinates on $S \subset \mathbb{C}$. Write $A$ in coordinates

$$A = \alpha + \phi \ ds + \psi \ dt,$$

$\alpha : S \rightarrow \mathcal{A}(\Sigma)$ defined by $\alpha(s, t) := A|_{\{(s, t)\} \times \Sigma}$

$\phi, \psi$ defined by contracting $A$ with $\partial_s, \partial_t$, respectively.
Local picture: The case $S \times \Sigma$

A connection $A$ on $Z$ is an **(ASD) instanton** if

\[ *_Z F_A = - F_A, \]

where $*_Z$ is the Hodge star on $Z$.

$A = \alpha + \phi \, ds + \psi \, dt$ is an instanton if and only if

\[
\begin{align*}
\partial_s \alpha + *_{\Sigma} \partial_t \alpha &= d_\alpha \phi + *_{\Sigma} d_\alpha \psi \\
F_\alpha &= *_{\Sigma}(\partial_t \phi - \partial_s \psi - [\phi, \psi])
\end{align*}
\]

where $*_{\Sigma}$ is the Hodge star on $\Sigma$. 
Local picture: The case $S \times \Sigma$

A connection $A$ on $Z$ is an **(ASD) instanton** if

$$\ast_Z F_A = -F_A,$$

where $\ast_Z$ is the Hodge star on $Z$.

$A = \alpha + \phi \, ds + \psi \, dt$ is an instanton if and only if

$$\partial_s \alpha + \ast_\Sigma \partial_t \alpha = d_\alpha \phi + \ast_\Sigma d_\alpha \psi$$

$$F_\alpha = \ast_\Sigma (\partial_t \phi - \partial_s \psi - [\phi, \psi])$$

where $\ast_\Sigma$ is the Hodge star on $\Sigma$. 
In particular, if we can ensure the curvature $F_\alpha$ is small, then

$$u := \Pi(\alpha) : S \rightarrow M(\Sigma)$$

is a well-defined function that is holomorphic. We are starting to see a map

$$\{\text{instantons}\} \rightarrow \{\text{holomorphic curves } S \rightarrow M(\Sigma)\},$$

but we need to ensure that $F_\alpha$ is small.
Local picture: The case $\mathbb{S} \times \Sigma$

In particular, if we can ensure the curvature $F_\alpha$ is small, then

$$u := \Pi(\alpha) : \mathbb{S} \rightarrow M(\Sigma)$$

is a well-defined function that is holomorphic. We are starting to see a map

$$\{\text{instantons}\} \rightarrow \{\text{holomorphic curves } \mathbb{S} \rightarrow M(\Sigma)\},$$

but we need to ensure that $F_\alpha$ is small.
Now assume we have a fiber bundle

\[ \Sigma \leftrightarrow Z \xrightarrow{F} S. \]
Global picture

Now assume we have a fiber bundle

\[ \Sigma \leftrightarrow Z^F \rightarrow S. \]
There exist perturbations that yield smooth moduli spaces (of instantons and holo. curves).

Assume everything is suitably perturbed.

The instanton moduli space

$$\{ A \in \mathcal{A}(Z) \mid \ast F_A = -F_A \} / G_0$$

is a smooth manifold.

We will denote the dimension near $[A]$ by $\text{dim}(A)$. 
Global picture

There exist perturbations that yield smooth moduli spaces (of instantons and holo. curves).

Assume everything is suitably perturbed.

The instanton moduli space

$$\{ A \in A(Z) \mid \star F_A = -F_A \} / G_0$$

is a smooth manifold.

We will denote the dimension near [A] by dim(A).
There exist perturbations that yield smooth moduli spaces (of instantons and holo. curves).

Assume everything is suitably perturbed.

The instanton moduli space

\[ \{ A \in \mathcal{A}(Z) \mid F_A = -F_A \} / G_0 \]

is a smooth manifold.

We will denote the dimension near \([A]\) by \(\text{dim}(A)\).
Global picture

There exist perturbations that yield smooth moduli spaces (of instantons and holo. curves).

Assume everything is suitably perturbed.

The instanton moduli space

\[ \{ A \in \mathcal{A}(Z) \mid * F_A = -F_A \} / \mathcal{G}_0 \]

is a smooth manifold.

We will denote the dimension near \([A]\) by \(\dim(A)\).
Global picture

Theorem (D. (2013))

Suppose $S$ is closed or has cylindrical ends. Given any $\epsilon_0 > 0$, there is a metric on $S$ so that if $A$ is an instanton on $Z$ with $\dim(A) \leq 3$, then

$$\sup_{x \in S} \| F_\alpha \|_{L^2(\Sigma_x)} \leq \epsilon_0.$$  

That is, $\alpha(x) = A|_{\Sigma_x}$ is in the domain of $\Pi$ for all $x \in S$. 
Global picture

Theorem (D. (2013))

Suppose $S$ is closed or has cylindrical ends. Given any $\epsilon_0 > 0$, there is a metric on $S$ so that if $A$ is an instanton on $Z$ with $\dim(A) \leq 3$, then

$$\sup_{x \in S} \|F_\alpha\|_{L^2(\Sigma_x)} \leq \epsilon_0.$$ 

That is, $\alpha(x) = A|_{\Sigma_x}$ is in the domain of $\Pi$ for all $x \in S$. 
Define

\( \mathcal{N} : \{ \text{instantons } A \text{ on } Z \text{ with } \dim(A) \leq 3 \} / \{ \text{gauge} \} \)

\[ \longrightarrow \{ \text{holomorphic sections } S \rightarrow M(Z) \} . \]

by sending \([A]\) to the map \(x \mapsto \Pi(\alpha(x))\).
Remarks

• The map $\mathcal{N}$ preserves the dimension of the moduli spaces.

• When $S$ has cylindrical ends then $\mathcal{N}$ preserves the broken trajectory compactifications of the moduli spaces.
Remarks

- The map $\mathcal{N}$ preserves the dimension of the moduli spaces.
- When $S$ has cylindrical ends then $\mathcal{N}$ preserves the broken trajectory compactifications of the moduli spaces.
Theorem (D.-McNamara (2014))

The map $N$ is a $C^1$-embedding.

Proof.

Injectivity of $N$: If $N([A_0]) = N([A_1])$, then $\Pi(\alpha_0) = \Pi(\alpha_1)$ as maps $S \to M(Z)$. This implies

$$\alpha_0 = g^* \alpha_1$$

for some complex gauge transformation $g : S \to G_0^C$. The instanton equation and irreducibility force $g$ to take values in the real gauge group $G_0$. This implies $A_0$ and $A_1$ lie in the same gauge orbit, so $[A_0] = [A_1]$.

Injectivity of the linearization $D_{[A]} N$ is similar.
Theorem (D.-McNamara (2014))

The map $\mathcal{N}$ is a $C^1$-embedding.

Proof.
Injectivity of $\mathcal{N}$: If $\mathcal{N}([A_0]) = \mathcal{N}([A_1])$, then $\Pi(\alpha_0) = \Pi(\alpha_1)$ as maps $S \rightarrow M(Z)$. This implies

$$\alpha_0 = g^* \alpha_1$$

for some complex gauge transformation $g : S \rightarrow G_0^\mathbb{C}$. The instanton equation and irreducibility force $g$ to take values in the real gauge group $G_0$. This implies $A_0$ and $A_1$ lie in the same gauge orbit, so $[A_0] = [A_1]$.

Injectivity of the linearization $D_{[A]} \mathcal{N}$ is similar.
Theorem (D.-McNamara (2014))

The map $\mathcal{N}$ is a $C^1$-embedding.

Proof.
Injectivity of $\mathcal{N}$: If $\mathcal{N}([A_0]) = \mathcal{N}([A_1])$, then $\Pi(\alpha_0) = \Pi(\alpha_1)$ as maps $S \to M(Z)$. This implies

$$\alpha_0 = g^* \alpha_1$$

for some complex gauge transformation $g : S \to G^C_0$. The instanton equation and irreducibility force $g$ to take values in the real gauge group $G_0$. This implies $A_0$ and $A_1$ lie in the same gauge orbit, so $[A_0] = [A_1]$.

Injectivity of the linearization $D_{[A]}\mathcal{N}$ is similar.
Theorem (D.-McNamara (2014))

The map $\mathcal{N}$ is a $C^1$-embedding.

Proof.

Injectivity of $\mathcal{N}$: If $\mathcal{N}([A_0]) = \mathcal{N}([A_1])$, then $\Pi(\alpha_0) = \Pi(\alpha_1)$ as maps $S \rightarrow M(Z)$. This implies

$$\alpha_0 = g^* \alpha_1$$

for some complex gauge transformation $g : S \rightarrow G_0^C$. The instanton equation and irreducibility force $g$ to take values in the real gauge group $G_0$. This implies $A_0$ and $A_1$ lie in the same gauge orbit, so $[A_0] = [A_1]$.

Injectivity of the linearization $D_{[A]}\mathcal{N}$ is similar.
Corollary (Dostoglou-Salamon, D.-McNamara, D.-)

The map $\mathcal{N}$ is a diffeomorphism.

Proof 1 (Kähler case).

If $Z$ is Kähler, then there is a complexified gauge group $G(Z)^C$. Any holomorphic section can be represented by a connection $A_0$ on $Z$ that is in the semistable range. Then there is a (unique up to real gauge transformation) instanton $A$ that is in the $G(Z)^C$-orbit of $A_0$. The map sending $[A_0]$ to $[A]$ is an approximate inverse of $\mathcal{N}$. 
Corollary (Dostoglou-Salamon, D.-McNamara, D.-)

The map $N$ is a diffeomorphism.

Proof 1 (Kähler case).

If $Z$ is Kähler, then there is a complexified gauge group $G(Z)^\mathbb{C}$. Any holomorphic section can be represented by a connection $A_0$ on $Z$ that is in the semistable range. Then there is a (unique up to real gauge transformation) instanton $A$ that is in the $G(Z)^\mathbb{C}$-orbit of $A_0$. The map sending $[A_0]$ to $[A]$ is an approximate inverse of $N$. 

□
Proof 2 (general case).

Given a connection $A_0$ that represents a holomorphic section of $M(Z)$, one can use an implicit function theorem as in Dostoglou-Salamon [DS3] to find a nearby instanton $A$. This provides an approximate inverse to $N$.

Proof 3 (general case).

Given a connection $A_0$ that represents a holomorphic section of $M(Z)$, one can use the Yang-Mills heat flow to flow down to an instanton $A$. This is another approximate inverse of $N$. 

□
Proof 2 (general case).

Given a connection $A_0$ that represents a holomorphic section of $M(Z)$, one can use an implicit function theorem as in Dostoglou-Salamon [DS3] to find a nearby instanton $A$. This provides an approximate inverse to $\mathcal{N}$. 

Proof 3 (general case).

Given a connection $A_0$ that represents a holomorphic section of $M(Z)$, one can use the Yang-Mills heat flow to flow down to an instanton $A$. This is another approximate inverse of $\mathcal{N}$. 

From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

$A(Z)$

instantons

holomorphic section representatives
More general 4-manifolds
Set-up (reminder)

Suppose

- $S$ is a connected, oriented surface with cylindrical ends,
- $Z$ is an oriented 4-manifold with the same number of cylindrical ends as $S$,
- $F : Z \to S$ preserves the ends and has connected, nonempty fibers, and
- $R \to Z$ is a principal $SO(3)$-bundle that restricts to the non-trivial bundle on some fiber.
Set-up (reminder)

Suppose

- $S$ is a connected, oriented surface with cylindrical ends,
- $Z$ is an oriented 4-manifold with the same number of cylindrical ends as $S$,
- $F : Z \to S$ preserves the ends and has connected, nonempty fibers, and
- $R \to Z$ is a principal $\text{SO}(3)$-bundle that restricts to the non-trivial bundle on some fiber.
Set-up (reminder)

Suppose

• $S$ is a connected, oriented surface with cylindrical ends,

• $Z$ is an oriented 4-manifold with the same number of cylindrical ends as $S$,

• $F : Z \to S$ preserves the ends and has connected, nonempty fibers, and

• $R \to Z$ is a principal $SO(3)$-bundle that restricts to the non-trivial bundle on some fiber.
Set-up (reminder)

Suppose

- $S$ is a connected, oriented surface with cylindrical ends,
- $Z$ is an oriented 4-manifold with the same number of cylindrical ends as $S$,
- $F : Z \to S$ preserves the ends and has connected, nonempty fibers, and
- $R \to Z$ is a principal $SO(3)$-bundle that restricts to the non-trivial bundle on some fiber.
History/Summary
Def. of \( N \)
Local picture
Global picture
More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

Set-up (reminder)

Suppose

- \( S \) is a connected, oriented surface with cylindrical ends,
- \( Z \) is an oriented 4-manifold with the same number of cylindrical ends as \( S \),
- \( F : Z \to S \) preserves the ends and has connected, nonempty fibers, and
- \( R \to Z \) is a principal \( \text{SO}(3) \)-bundle that restricts to the non-trivial bundle on some fiber.
Compare instantons on $Z$ with certain symplectic objects, and try to extend the results above.

Instantons are well-defined, and the (relative) Donaldson invariant depends only on $Z$, the topological type of the bundle $R$, and the homotopy class of $F$.

The relevant symplectic objects are holomorphic quilts. These were defined by Wehrheim-Woodward [WW2], and one uses Morse 2-functions to obtain these from 4-manifolds.
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary
Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds
Morse 2-functions
Quilts
Quilts from instantons

Idea

Compare instantons on $Z$ with certain symplectic objects, and try to extend the results above.

Instantons are well-defined, and the (relative) Donaldson invariant depends only on $Z$, the topological type of the bundle $R$, and the homotopy class of $F$.

The relevant symplectic objects are holomorphic quilts. These were defined by Wehrheim-Woodward [WW2], and one uses Morse 2-functions to obtain these from 4-manifolds.
Compare instantons on $Z$ with certain symplectic objects, and try to extend the results above.

Instantons are well-defined, and the (relative) Donaldson invariant depends only on $Z$, the topological type of the bundle $R$, and the homotopy class of $F$.

The relevant symplectic objects are holomorphic quilts. These were defined by Wehrheim-Woodward [WW2], and one uses Morse 2-functions to obtain these from 4-manifolds.
Compare instantons on $Z$ with certain symplectic objects, and try to extend the results above.

Instantons are well-defined, and the (relative) Donaldson invariant depends only on $Z$, the topological type of the bundle $R$, and the homotopy class of $F$.

The relevant symplectic objects are holomorphic quilts. These were defined by Wehrheim-Woodward [WW2], and one uses Morse 2-functions to obtain these from 4-manifolds.
Morse 2-functions

$F : Z \to S$ is a **Morse 2-function** if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in $S$ lies in a neighborhood $I \times I \subset S$, with $I = [0, 1]$, and this neighborhood satisfies

1. $F^{-1}(I \times I) = I \times Y$ where $Y$ is a 3-manifold,
2. in these coordinates
   \[
   F(s, y) = (s, f_s(y)),
   \]
   where $f_s : Y \to I$ is Morse for all but a finite set of exceptional times $s$, and
3. each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

$F: Z \to S$ is a **Morse 2-function** if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in $S$ lies in a neighborhood $I \times I \subset S$, with $I = [0, 1]$, and this neighborhood satisfies

- $F^{-1}(I \times I) = I \times Y$ where $Y$ is a 3-manifold,
- in these coordinates

$$F(s, y) = (s, f_s(y)),$$

where $f_s : Y \to I$ is Morse for all but a finite set of exceptional times $s$, and
- each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

\( F : Z \to S \) is a **Morse 2-function** if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in \( S \) lies in a neighborhood \( I \times I \subset S \), with \( I = [0, 1] \), and this neighborhood satisfies

- \( F^{-1}(I \times I) = I \times Y \) where \( Y \) is a 3-manifold,
- in these coordinates
  \[
  F(s, y) = (s, f_s(y)),
  \]
  where \( f_s : Y \to I \) is Morse for all but a finite set of exceptional times \( s \), and
- each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

$F : Z \to S$ is a **Morse 2-function** if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in $S$ lies in a neighborhood $I \times I \subset S$, with $I = [0, 1]$, and this neighborhood satisfies

- $F^{-1}(I \times I) = I \times Y$ where $Y$ is a 3-manifold,
- in these coordinates

$$F(s, y) = (s, f_s(y)),$$

where $f_s : Y \to I$ is Morse for all but a finite set of exceptional times $s$, and
- each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

$F : Z \to S$ is a Morse 2-function if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in $S$ lies in a neighborhood $I \times I \subset S$, with $I = [0, 1]$, and this neighborhood satisfies

- $F^{-1}(I \times I) = I \times Y$ where $Y$ is a 3-manifold,
- in these coordinates

$$F(s, y) = (s, f_s(y)),$$

where $f_s : Y \to I$ is Morse for all but a finite set of exceptional times $s$, and

- each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

\( F : Z \to S \) is a **Morse 2-function** if it can be written locally as a ‘homotopy of Morse functions’. That is, if each point in \( S \) lies in a neighborhood \( I \times I \subset S \), with \( I = [0, 1] \), and this neighborhood satisfies

- \( F^{-1}(I \times I) = I \times Y \) where \( Y \) is a 3-manifold,
- in these coordinates
  \[
  F(s, y) = (s, f_s(y)),
  \]
  where \( f_s : Y \to I \) is Morse for all but a finite set of exceptional times \( s \), and
- each exceptional time corresponds to a critical point birth, death or crossing.
Morse 2-functions

Example 1: Suppose $F : Y \to S$ is a fiber bundle.

Example 2: Suppose $Y$ is a closed 3-manifold and $f : Y \to S^1$ is a Morse function. Consider $Z = \mathbb{R} \times Y$ and $S = \mathbb{R} \times S^1$. Then $F(s, y) = (s, f(y))$ is a Morse 2-function. This is the cylindrical case.

In general, if $Z$ has cylindrical ends, then we assume that any Morse 2-function restricts to the cylindrical case on the ends.
Morse 2-functions

Example 1: Suppose $F : Y \to S$ is a fiber bundle.

Example 2: Suppose $Y$ is a closed 3-manifold and $f : Y \to S^1$ is a Morse function. Consider $Z = \mathbb{R} \times Y$ and $S = \mathbb{R} \times S^1$. Then $F(s, y) = (s, f(y))$ is a Morse 2-function. This is the cylindrical case.

In general, if $Z$ has cylindrical ends, then we assume that any Morse 2-function restricts to the cylindrical case on the ends.
Morse 2-functions

Example 1: Suppose $F : Y \to S$ is a fiber bundle.

Example 2: Suppose $Y$ is a closed 3-manifold and $f : Y \to S^1$ is a Morse function. Consider $Z = \mathbb{R} \times Y$ and $S = \mathbb{R} \times S^1$. Then $F(s, y) = (s, f(y))$ is a Morse 2-function. This is the cylindrical case.

In general, if $Z$ has cylindrical ends, then we assume that any Morse 2-function restricts to the cylindrical case on the ends.
Morse 2-functions

It is useful to trace out the critical values of the $f_s$ in the surface $S$. This is called the Cerf graphic. The critical values form the seams, and the regular values form the patches.
Morse 2-functions

In the cylindrical case \( Z = \mathbb{R} \times Y \), the Cerf graphic takes the form:

\[
S = \mathbb{R} \times S^1
\]
Morse 2-functions

Here is the Cerf graphic for some Morse 2-function \( F : Z \rightarrow S \), where \( S \) is a punctured genus 3 surface.
Morse 2-functions

Here is the Cerf graphic for some Morse 2-function $F : Z \rightarrow S$, where $S$ is a punctured genus 3 surface.
Morse 2-functions
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

Morse 2-functions
Morse 2-functions

\[ \Sigma_x \]

\[ \star \]

\[ x \]
Morse 2-functions

$\Sigma_x$

$\star$
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $N$

Local picture

Global picture

More general 4-manifolds

Morse 2-functions

Quilts

Quilts from instantons

Morse 2-functions

$\Sigma_x$
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

$\Sigma_x$

Morse 2-functions
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

**Morse 2-functions**

![Diagram](image-url)
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

Morse 2-functions

$\Sigma_x$
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

Morse 2-functions

\[ \Sigma_x \]
Morse 2-functions

\[ \Sigma_x \]
From instantons to quilts with seam degenerations

David L. Duncan

Morse 2-functions

\[ \Sigma_x \]
From instantons to quilts with seam degenerations

David L. Duncan

History/Summary

Def. of $\mathcal{N}$
Local picture
Global picture

More general 4-manifolds

Morse 2-functions
Quilts
Quilts from instantons

Morse 2-functions

$\Sigma_x$
Morse 2-functions

Theorem (Gay-Kirby (2011))

Suppose $F : Z \to S$ is any smooth function with connected and non-empty fibers. Then $F$ is homotopic to a Morse 2-function that has connected and non-empty fibers. This Morse 2-function is unique up to certain homotopies of Morse 2-functions.
Quilts from Morse 2-functions

The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \to S$ is a Morse 2-function.

Consider the surface $S$ with the diagram coming from the critical points.

Each point $x$ in a patch is a regular point for $F$. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$. 
The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \to S$ is a Morse 2-function.

Consider the surface $S$ with the diagram coming from the critical points.

Each point $x$ in a patch is a regular point for $F$. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$. 
Quilts from Morse 2-functions

The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \rightarrow S$ is a Morse 2-function.

Consider the surface $S$ with the diagram coming from the critical points.

Each point $x$ in a patch is a regular point for $F$. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$. 
Quilts from Morse 2-functions

The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \to S$ is a Morse 2-function.

Consider the surface $S$ with the diagram coming from the critical points.

Each point $x$ in a patch is a regular point for $F$. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$. 
Quilts from Morse 2-functions

The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \to S$ is a Morse 2-function.

Consider the surface $S$ with the diagram coming from the critical points.

Each point $x$ in a patch is a regular point for $F$. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$. 
Quilts from Morse 2-functions
Quilts from Morse 2-functions
Quilts from Morse 2-functions

$M(\Sigma_x)$
Quilts from Morse 2-functions

Over each patch $S_i$ is a fiber bundle, as before: $M(Z_i) \rightarrow S_i$

Think of the singular fibers as interpolating between non-singular fibers. (In symplectic language, there is a Lagrangian correspondence here.)

A quilt is a collection $\{u_i\}$ where $u_i$ is a section $S_i \rightarrow M(Z_i)$, and we require that these agree on the seams.

The quilt is holomorphic if each $u_i$ is holomorphic with respect to some complex structure.
Quilts from Morse 2-functions

Over each patch $S_i$ is a fiber bundle, as before: $M(Z_i) \to S_i$

Think of the singular fibers as interpolating between non-singular fibers. (In symplectic language, there is a *Lagrangian correspondence* here.)

A **quilt** is a collection $\{u_i\}$ where $u_i$ is a section $S_i \to M(Z_i)$, and we require that these agree on the seams.

The quilt is **holomorphic** if each $u_i$ is holomorphic with respect to some complex structure.
Quilts from Morse 2-functions

Over each patch $S_i$ is a fiber bundle, as before: $M(Z_i) \to S_i$

Think of the singular fibers as interpolating between non-singular fibers. (In symplectic language, there is a Lagrangian correspondence here.)

A **quilt** is a collection $\{u_i\}$ where $u_i$ is a section $S_i \to M(Z_i)$, and we require that these agree on the seams.

The quilt is **holomorphic** if each $u_i$ is holomorphic with respect to some complex structure.
Quilts from Morse 2-functions

Over each patch $S_i$ is a fiber bundle, as before: $M(Z_i) \to S_i$

Think of the singular fibers as interpolating between non-singular fibers. (In symplectic language, there is a Lagrangian correspondence here.)

A quilt is a collection $\{u_i\}$ where $u_i$ is a section $S_i \to M(Z_i)$, and we require that these agree on the seams.

The quilt is holomorphic if each $u_i$ is holomorphic with respect to some complex structure.
Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of $R$, and the homotopy class of $f$.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of $F$. Establishing this is an active research project of Wehrheim.)
Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of $R$, and the homotopy class of $f$.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of $F$. Establishing this is an active research project of Wehrheim.)
Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of $R$, and the homotopy class of $f$.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of $F$. Establishing this is an active research project of Wehrheim.)
Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of $R$, and the homotopy class of $f$.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of $F$. Establishing this is an active research project of Wehrheim.)
Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of $R$, and the homotopy class of $f$.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of $F$. Establishing this is an active research project of Wehrheim.)
Suppose $A$ is an instanton. For each patch $S_i$ and $x \in S_i$, set 
\[
\alpha_i(x) = A|\{F^{-1}(x)\}.
\]
As before, there is a metric on $Z$ for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \Pi(\alpha_i(x))$. So $u_i$ is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that $u_i$ is holomorphic in the interior of the patch $S_i$. 
Suppose $A$ is an instanton. For each patch $S_i$ and $x \in S_i$, set $\alpha_i(x) = A|_{\{F^{-1}(x)\}}$.

As before, there is a metric on $Z$ for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \prod(\alpha_i(x))$. So $u_i$ is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that $u_i$ is holomorphic in the interior of the patch $S_i$. 
Suppose $A$ is an instanton. For each patch $S_i$ and $x \in S_i$, set

$$\alpha_i(x) = A|_{\{ F^{-1}(x) \}}.$$ 

As before, there is a metric on $Z$ for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \Pi(\alpha_i(x))$. So $u_i$ is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that $u_i$ is holomorphic in the interior of the patch $S_i$. 
Suppose $A$ is an instanton. For each patch $S_i$ and $x \in S_i$, set $\alpha_i(x) = A|_{\{F^{-1}(x)\}}$.

As before, there is a metric on $Z$ for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \prod(\alpha_i(x))$. So $u_i$ is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that $u_i$ is holomorphic in the interior of the patch $S_i$. 
From instantons to quilts

Suppose $A$ is an instanton. For each patch $S_i$ and $x \in S_i$, set $\alpha_i(x) = A|_{\{F^{-1}(x)\}}$.

As before, there is a metric on $Z$ for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \prod(\alpha_i(x))$. So $u_i$ is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that $u_i$ is holomorphic in the interior of the patch $S_i$. 
From instantons to quilts

That is, just as before, we should get a map from the moduli space of instantons to the moduli space of holomorphic quilts.

The catch is that the relevant metric on each fiber $\Sigma_x$ becomes singular (blows up) as $x$ approaches a seam. This reflects the fact that the fibers over the seams are singular.
From instantons to quilts

That is, just as before, we should get a map from the moduli space of instantons to the moduli space of holomorphic quilts.

The catch is that the relevant metric on each fiber $\Sigma_x$ becomes singular (blows up) as $x$ approaches a seam. This reflects the fact that the fibers over the seams are singular.
Questions/Future directions

Q1: Do the $u_i$ actually have the expected boundary conditions? This should be the case. The work is in studying how the map $\Pi : \mathcal{A}^{ss}(\Sigma) \to M(\Sigma)$ behaves under a degeneration of the metric.

If this is the case, then the relevant holomorphic quilts would be holomorphic with respect to a complex structure that becomes singular near the boundary.
Questions/Future directions

Q1: Do the $u_i$ actually have the expected boundary conditions?

This should be the case. The work is in studying how the map $\Pi : \mathcal{A}^{ss}(\Sigma) \rightarrow M(\Sigma)$ behaves under a degeneration of the metric.

If this is the case, then the relevant holomorphic quilts would be holomorphic with respect to a complex structure that becomes singular near the boundary.
Questions/Future directions

Q1: Do the $u_i$ actually have the expected boundary conditions? This should be the case. The work is in studying how the map $\Pi : A^{ss}(\Sigma) \to M(\Sigma)$ behaves under a degeneration of the metric.

If this is the case, then the relevant holomorphic quilts would be holomorphic with respect to a complex structure that becomes singular near the boundary.
Questions/Future directions

Q1: Do the $u_i$ actually have the expected boundary conditions?
This should be the case. The work is in studying how the map
$\Pi : \mathcal{A}^{ss}(\Sigma) \to M(\Sigma)$ behaves under a degeneration of the metric.

If this is the case, then the relevant holomorphic quilts would be holomorphic with respect to a complex structure that becomes singular near the boundary.
Questions/Future directions

Q2: Does the associated moduli space have the regularity/compactness properties needed to obtain a well-defined invariant?
Questions/Future directions

Supposing these questions have favorable answers, then we should be able to extend the analysis from the fiber bundle picture above to show that the induced map $\mathcal{N}$ is a diffeomorphism from the moduli space of instantons to the moduli space of holomorphic quilts.

Even if these questions do not have favorable answers, there is already enough going on here to obtain strong results. For example, these ideas were used to prove a compactness theorem [D1] that studies the behavior of instantons under certain degenerations of the underlying metric.
Questions/Future directions

Supposing these questions have favorable answers, then we should be able to extend the analysis from the fiber bundle picture above to show that the induced map $\mathcal{N}$ is a diffeomorphism from the moduli space of instantons to the moduli space of holomorphic quilts.

Even if these questions do not have favorable answers, there is already enough going on here to obtain strong results. For example, these ideas were used to prove a compactness theorem [D1] that studies the behavior of instantons under certain degenerations of the underlying metric.
Thank you for your attention.


From instantons to quilts with seam degenerations

David L. Duncan

Bibliography


