1. (a) Apply A3 with \(c = a\) to get \(a^2 \leq ab\). Since \(0 \leq a\) and \(a \leq b\), A1 implies that \(0 \leq b\). This means that we can apply A3 again but with \(c = b\) to get \(ab \leq b^2\). By A1 we know that \(a^2 \leq ab\) and \(ab \leq b^2\) implies \(a^2 \leq b^2\), as desired.

(b) We have \(a \leq 0\), so by A2 we can add \(-a\) to both sides to get \(0 \leq -a\).

(c) By the previous problem we have \(0 \leq -a\), so we can apply A3 with \(c = -a\) to get \(-b^2 \leq -ba\). By A1 we know that \(b \leq 0\), so by the previous problem we get \(0 \leq -b\). Now apply A3 with \(c = -b\) to get \(-ba \leq -a^2\), and hence \(-b^2 \leq -a^2\). By A2 we can add \(a^2\) and \(b^2\) to both sides to get \(a^2 \leq b^2\), as desired.

(d) We prove this by cases. If \(0 \leq b\), then apply the Problem (a) to get \(0^2 \leq b^2\). Since \(0^2 = 0\), we are done in this case. If \(b \leq 0\), then apply the third problem to get \(0 \leq b^2\).

(e) By the previous problem we know \(0 \leq (a - b)^2\). Expanding out the right-hand side gives \(0 \leq a^2 + b^2 - 2ab\). Apply A2 with \(c = 2ab\) to get \(2ab \leq a^2 + b^2\). Now apply A3 with \(c = 1/2\).

(f) Apply the previous problem replacing \(a\) with \(\delta a\) and \(b\) with \(\delta^{-1}b\). Notice that since \(\delta \neq 0\), we have that \(\delta^{-1}\) is a real number, and so the previous problem applies.

(g) If \(a = b\), then \(ab = a^2\) and \(\frac{1}{2}(a^2 + b^2) = a^2\) as well. Conversely, suppose \(ab = \frac{1}{2}(a^2 + b^2)\). This gives \(2ab = a^2 + b^2\), and hence \(0 = a^2 - 2ab + b^2\). The right-hand side factors to give \(0 = (a - b)^2\). Taking the square root gives \(0 = a - b\), which implies \(a = b\).

(h) Apply Problem (e), but replace \(a\) with \(\sqrt{a}\) and \(b\) with \(\sqrt{b}\). Notice that since \(a\) and \(b\) are non-negative, their square roots are real numbers and so Problem (e) does indeed apply.

2. We prove this using contrapositive. So assume that \(x\) and \(y\) do not have the same parity, and we want to show that \(3x + 5y\) is odd. Since \(x\) and \(y\) do not have the same parity, there are two cases to consider.

Case 1: \(x\) is odd and \(y\) is even.
In this case we can write $x = 2n + 1$ and $y = 2m$ for some integers $n$ and $m$. Then

$$3x + 5y = 3(2n + 1) + 5(2m) = 6n + 3 + 10m = 2(3n + 5m + 1) + 1.$$  

This is odd, so we are done in this case.

**Case 2: $x$ is even and $y$ is odd.**

In this case we can write $x = 2n$ and $y = 2m + 1$ for some integers $n$ and $m$. Then

$$3x + 5y = 3(2n) + 5(2m + 1) = 6n + 10m + 5 = 2(3n + 5m + 2) + 1.$$  

This is odd, so we are done in this case as well.

3. To prove the statement, we will use the following lemma.

**Lemma.** If $z \in \mathbb{R}$, then $-z \leq |z|$.

**Proof.** We prove the lemma using cases. If $z \geq 0$, then $-z \leq 0$ and $z = |z|$. Then $-z \leq 0 \leq z = |z|$, so we are done in this case. If $z \leq 0$, then $|z| = -z$ and so we are done in this case as well.

Now to prove the statement, assume $x + y < 0$. Then $|x + y| = -x + (-y)$. By the lemma we have $-x \leq |x|$ and $-y \leq |y|$, so

$$|x + y| \leq |x| + |y|,$$

as desired.