1. Decide whether the following statements are true or false. Prove the true ones and provide counterexamples for the false ones.

(a) If \( a_n \) converges, then \( a_n/n \) converges.

(b) If \( a_n \) converges and \( b_n \) is bounded, then \( a_n b_n \) converges.

(c) If \( a_n \to \infty \) and \( b_n \to -\infty \) as \( n \to \infty \), then \( a_n + b_n \to 0 \) as \( n \to \infty \).

(d) If \( a_n \to 0 \) and \( b_n \to 1 \) as \( n \to \infty \), then \( b_n/a_n \to \infty \) as \( n \to \infty \).

2. Suppose that \( \{a_n\} \) is bounded. Prove that \( a_n/n^k \to 0 \), as \( n \to \infty \) for all \( k \in \mathbb{N} \).

3. Using the formal definition of the limit proof that if \( \lim_{n \to \infty} a_n = 1 \) then \( \lim_{n \to \infty} \frac{a_n^2 - e}{a_n} = 1 - e \).

4. (AC) Let \( S \) be the set of all functions \( f : \mathbb{N} \to \mathbb{N} \). Define a relation on \( S \) by letting \( f \sim g \) if and only if \( f(n) = g(n) \) for infinitely many \( n \). Is this an equivalence relation? If so describe the equivalence classes.

5. (AC) Prove (assuming basic results of calculus) that \( \int_{0}^{\infty} x^n e^{-x} dx = n! \).

6. (AC) For a function \( f : \mathbb{R} \to \mathbb{R} \), define \( \lim_{x \to c} f(x) = L \) to mean that \( \forall \epsilon > 0 \ \exists \delta > 0 \) such that \( \forall x \in \mathbb{R}, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon \). Define the function \( s : \mathbb{R} \to \mathbb{R} \) by

\[
s(x) = \begin{cases} 
0 & : x \leq 0 \\
1 & : x > 0 
\end{cases}
\]

Prove by negating the definition of limit that it is not true that \( \lim_{x \to 0} s(x) = 0 \).

7. (a) Use a multiplication table to find all values \( a \in \mathbb{Z}_7 \) for which the equation

\[
x^2 = a
\]

has a solution \( x \in \mathbb{Z}_7 \). For each such \( a \), list all of the solutions \( x \).

(b) Find all solutions \( x \in \mathbb{Z}_7 \) to the equation \( x^2 + 2x + 6 = 0 \).

8. Use quantifiers to express what it means for a sequence \( (x_n)_{n \in \mathbb{N}} \) to diverge. You cannot use the terms not or converge.

9. Suppose \( A, B \subseteq \mathbb{R} \) are bounded and non-empty. Show that \( \sup(A \cup B) = \max\{\sup(A), \sup(B)\} \).

10. Suppose \( S \subseteq \mathbb{R} \) is bounded and non-empty. Define a new set \( 3S \) by \( 3S = \{3x \mid x \in S\} \). Show that \( \sup(3S) = 3\sup(S) \).

11. Let \( A, B \) be sets, and suppose there is a surjection \( f : A \to B \). Prove that there is an injection \( g : B \to A \).
12. (a) Define \( x \in \mathbb{R} \) to be a **linear algebraic number** if there are integers \( a, b \in \mathbb{Z} \), with \( a \neq 0 \), such that \( ax + b = 0 \). 
Prove that the set of linear algebraic numbers is countable. *Hint: Construct an injection into the set \( \mathbb{Z}^2 \).*

(b) Define \( x \in \mathbb{R} \) to be a **quadratic algebraic number** if there are integers \( a, b, c \in \mathbb{Z} \), with \( a \neq 0 \), such that \( ax^2 + bx + c = 0 \). Prove that the set of quadratic algebraic numbers is countable. *Hint: Construct an injection into the set \( \mathbb{Z}^3 \).*

13. Use the formal definition of limit to prove the following.

\[
\begin{align*}
(a) & \quad \lim_{n \to \infty} \frac{n^2 + 3}{2n^3 - 4} = 0 \\
(b) & \quad \lim_{n \to \infty} \frac{4n - 5}{2n + 7} = 2 \\
(c) & \quad \lim_{n \to \infty} \frac{n^3 - 3n}{n + 5} = +\infty \\
(d) & \quad \lim_{n \to \infty} \frac{n^2 - 7}{1 - n} = -\infty 
\end{align*}
\]

14. For each of the following, determine if \( \sim \) defines an equivalence relation on the set \( S \). If it does, prove it and describe the equivalence classes. If it does not, explain why.

(a) \( S = \mathbb{R} \times \mathbb{R} \). For \((a, b)\) and \((c, d)\) \( \in \) \( S \), define \((a, b) \sim (c, d)\) if \( 3a + 5b = 3c + 5d \).

(b) \( S = \mathbb{R} \). For \( a, b \in S \), \( a \sim b \) if \( a < b \).

(c) \( S = \mathbb{Z} \). For \( a, b \in S \), \( a \sim b \) if \( a \mid b \).

(d) \( S = \mathbb{R} \times \mathbb{R} \). For \((a, b)\) and \((c, d)\) \( \in \) \( S \), define \((a, b) \sim (c, d)\) if \( \lceil a \rceil = \lceil c \rceil \) and \( \lceil b \rceil = \lceil d \rceil \). Here \( \lceil x \rceil \) is the smallest integer greater than or equal to \( x \).

15. Consider \( \mathbb{Z}_n \).

(a) Under what conditions on \( n \) does every nonzero element have a multiplicative inverse? How about an additive inverse?

(b) Does every nonzero element have a multiplicative inverse in \( \mathbb{Z}_{21} \)?

(c) Does 5 have a multiplicative inverse in \( \mathbb{Z}_{21} \)? Explain why or why not. If it does, find \( 5^{-1} \).

(d) Solve the equation \( 5x - 14 = 19 \) in \( \mathbb{Z}_{21} \).

16. Let \( A = \{a, b, c\} \) and \( B = \{a, x\} \). List all elements of

(a) \( A \cup B \)

(b) \( A \cap B \)

(c) \( A \setminus B \)
17. Let \( S(n) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \max\{x, y\} = n\} \). Prove that \( S(3) \cap S(5) \) is the empty set.

18. Let \( A \) and \( B \) be sets with \( n \) elements. Show that any injective function from \( A \) to \( B \) is surjective as well using induction on \( n \).

19. Let \( f : \mathbb{N} \to \mathbb{N} \), given by \( f(n) = |n - 4| \).
   (a) Prove that \( f \) is surjective
   (b) Prove that \( f \) is not injective

20. Let \( f : A \to B \) and \( g : B \to A \) be functions satisfying \( f(g(x)) = x \) for all \( x \in B \). Prove that \( f \) is surjective.

21. Let \( X \) be a set with \( n \) elements and \( B = \{p, q\} \). Find the number of surjective functions from \( X \) to \( B \).

22. Describe a concrete bijection from \( \mathbb{N} \) to \( \mathbb{N} \times \{1, 2, 3\} \). Briefly tell why it is injective and surjective.

23. Make a truth table for \( \neg (A \lor B) \implies A \land B \). Find a shorter logically equivalent expression.

24. Find the negations of the following statements:
   (a) \( (A \lor B) \land (B \lor C) \)
   (b) \( A \implies (B \land C) \)
   (c) \( \forall x \exists y \ (P(x) \lor \neg Q(y)) \)