Meanings

- **Definition**: an explanation of the mathematical meaning of a word.
- **Theorem**: A statement that has been proven to be true.
- **Proposition**: A less important but nonetheless interesting true statement.
- **Lemma**: A true statement used in proving other true statements (that is, a less important theorem that is helpful in the proof of other results).
- **Corollary**: A true statement that is a simple deduction from a theorem or proposition.
- **Proof**: The explanation of why a statement is true.
- **Conjecture**: A statement believed to be true, but for which we have no proof. (a statement that is being proposed to be a true statement).
- **Axiom**: A basic assumption about a mathematical situation. (a statement we assume to be true).

Examples

- **Definition 6.1**: A *statement* is a sentence that is either true or false— but not both. ([H], Page 53).
- **Theorem 10.1**: \( \mathbb{N} \), considered as a subset of \( \mathbb{R} \), is not bounded above. ([B], Page 96).
- **Corollary 10.2**: \( \mathbb{Z} \) is not bounded above. ([B], Page 96).
- **Proposition 10.4**: For each \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \). ([B], Page 96).
- **Lemma**: Lemmas are considered to be less important than propositions. But the distinction between categories is rather blurred. There is no formal distinction among a lemma, a proposition, and a theorem.
- **Axioms**: If \( m \) and \( n \) are integers, then \( m + n = n + m \). (Read [B] Page 4.)
- **Conjecture**: Mathematicians are making, testing and refining conjectures as they do their research.
Group Axioms

A Group is a set \( G \) together with an operation \( \# \), for which the following axioms are satisfied.

\begin{align*}
A_1. \text{ Closure: } & \forall a, b \in G, \ a \# b \in G \\
A_2. \text{ Associativity: } & \forall a, b, c \in G, \ (a \# b) \# c = a \# (b \# c) \\
A_3. \text{ Identity element: } & \exists e \in G \text{ such that } \forall a \in G, \ a \# e = e \# a = a \\
A_4. \text{ Inverse element: } & \forall a \in G, \ \exists b \in G \text{ such that } a \# b = b \# a = e
\end{align*}

1. Is \( \mathbb{N} \) with + a group?

2. Is \( \mathbb{Z} \) with + a group?

3. Do the axioms imply that if \( G \) is a group and \( a, b \in G \) then \( a \# b = b \# a \)?

4. Can you give an example of a group (all axioms \( A_1 - A_4 \) are satisfied) whose elements do not commute with each other?
Group Theorems

**Theorem:** The identity element is unique.

**Proof:**

**Theorem:** For every element $a \in G$ there exists a unique inverse.

**Proof:**
Axiomatic system 1:

Undefined terms: member, committee

A1. Every committee is a collection of at least two members.
A2. Every member is on at least one committee.

1. Find two different models for this set of axioms.

2. Discuss how it can be made categorical (there is a one-to-one correspondence between the elements in the model that preserves their relationship).

Axiomatic system 2:

Definition: A line $\ell$ intersects a line $m$ if there is a point $A$ that lies on both $\ell$ and $m$.

Undefined terms: point, line

A1. Every line is a set of at least two points.
A2. Each two lines intersect in a unique point.
A3. There are precisely three lines.

Find two different models for this set of axioms.
Axiomatic system 3:
Undefined terms: point, line, lie on

**Definition:** A line \( \ell \) **passes through** points \( A \) and \( B \) if \( A \) and \( B \) lie on \( \ell \).

**Definition:** A line \( \ell \) **intersects** a line \( m \) if there is a point \( A \) that lies on both \( \ell \) and \( m \).

\( A_1 \). There are exactly five points.
\( A_2 \). Exactly two points lie on each line.
\( A_3 \). At most one line passes through any two points.
\( A_4 \). There are exactly three lines.

Find two different models for this set of axioms.

**Theorem:** At least one pair of lines intersect.

**Conjecture:** There is a point that doesn’t lie on any line.