1. Suppose $V$ is an $n$-dimensional vector space equipped with an inner product $(\cdot, \cdot)$. Let $A : V \to V$ be a linear transformation that is symmetric; this means that

$$(Av, w) = (v, Aw)$$

for all $v, w \in V$. It is a theorem (you do not need to prove this), that there is a basis $e_1, \ldots, e_n$ consisting of eigenvectors of $A$. Let $\lambda_i$ be the eigenvalue of $e_i$.

(a) Show that if $\lambda_i \neq \lambda_j$, then

$$(e_i, e_j) = 0,$$

meaning that $e_i$ and $e_j$ are orthogonal. Hint: Begin by writing $\lambda_i(e_i, e_j) = (Ae_i, e_j)$.

(b) Assume all eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. Let $v \in V$. Since the $e_i$ form a basis, there are real numbers $a_1, \ldots, a_n$ so that

$$v = a_1 e_1 + \ldots + a_n e_n.$$  

Show that

$$a_i = \frac{1}{\|e_i\|^2} (v, e_i),$$

where $\|w\|^2 := (w, w)$ is the square of the length of $w$. Hint: Take the inner product of $v$ with $e_i$, and then use part (a).

2. Let $V$ be the set of functions $f : [0, \pi] \to \mathbb{R}$ such that

$$f(0) = 0, \quad f(\pi) = 0$$

and $f$ can be differentiated an infinite number of times. Equip $V$ with the inner product given by

$$(f, g) := \int_0^{\pi} f(x)g(x) \, dx.$$ 

(a) Let $A = d^2/dx^2$, and view this as a linear transformation on $V$. Show that $A$ is symmetric. Hint: By definition we have

$$(Af, g) = \int_0^{\pi} f''(x)g(x) \, dx.$$ 

Now integrate by parts twice.
(b) Suppose \( n, m \) are non-negative integers and \( n \neq m \). Show that
\[
\langle \sin(nx), \sin(mx) \rangle = 0.
\]

(c) Show that
\[
\| \sin(nx) \|^2 = \pi/2,
\]
where \( \| \cdot \|^2 = (\cdot, \cdot) \).

(d) Let \( f \in V \). It turns out this can be written as
\[
f(x) = \sum_n a_n \sin(nx)
\]
for some real numbers \( a_n \). Use Problem 1 to explain why we expect
\[
a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) \, dx.
\] (1)

(e) Consider the function
\[
f(x) = \begin{cases} 
x & \text{if } 0 \leq x \leq \pi/2 \\
\pi - x & \text{if } \pi/2 < x \leq \pi
\end{cases}
\]
Define \( a_n \) as in (1). Show that
\[
a_n = \begin{cases} 
0 & \text{if } n = 2m \text{ is even} \\
(-1)^m \frac{4}{\pi(2m+1)^2} & \text{if } n = 2m + 1 \text{ is odd}
\end{cases}
\]