Math 330 - Quiz 3 Study Guide

Choose four of the following problems to hand in on Wednesday, July 1.

1. Define a function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ by

$$g(z) = \frac{1}{\bar{z}}.$$  
(This has a conjugation.)

(a) Consider the triangle with vertices

$$A = 1, \quad B = i, \quad C = 1 + i.$$  
Find the image of $\triangle ABC$ under $g$.

(b) Consider the triangle with vertices

$$A = 1, \quad B = i, \quad D = 0.$$  
Find the image of the punctured triangle $\triangle ABD \setminus \{D\}$ under $g$. (Why was it necessary to puncture the triangle here?)

(c) Does $g$ preserve or reverse orientation?

2. (a) Consider a plane $P$ in $\mathbb{R}^3$. Define the term reflection over $P$. Hint: Mimic the definition of a reflection over a line in $\mathbb{R}^2$. Your answer should be described as a function $\mathbb{R}^3 \to \mathbb{R}^3$.

(b) Let $\mathbb{R}^2$ be the $xy$-plane in $\mathbb{R}^3$, and $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. What is the image of $S^2$ under the reflection over $\mathbb{R}^2$?

(c) Describe the set

$$\left\{ P \cap S^2 \mid P \text{ is a plane in } \mathbb{R}^3 \right\},$$

of possible intersections of $S^2$ with a plane.

(d) Let $C$ be a circle in $S^2$. What is the image of $C$ under the reflection over $\mathbb{R}^2$? Hint: Use parts (b) and (c).
3. (a) Consider the function
\[
f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f(z) = \begin{cases} \frac{az+b}{cz+d}, & z \neq -d/c, \Omega \\ \Omega, & z = -d/c \\ a/c, & z = \Omega \end{cases}
\]
and assume for simplicity that \(c \neq 0\). Show that \(f\) is a bijection if and only if \(ad - bc \neq 0\).

(b) Extend part (a) to the case where \(c = 0\) by defining the appropriate piecewise function \(f\) in this case.

4. Show that the set of Möbius transformations forms a group under composition.

5. Consider the line \(\ell \subset \mathbb{C}\) described by
\[
\ell = \left\{ i + e^{\pi/4}t \mid t \in \mathbb{R} \right\}.
\]

(a) Sketch \(\ell\).

(b) Find a formula for the function given by reflection over line \(\ell\). (This should involve conjugation in some way.)

(c) Find \(w \in \mathbb{C}\) and \(r \in \mathbb{R}\) so that
\[
\ell = \left\{ z \in \mathbb{C} \mid wz + \overline{wz} + r = 0 \right\}.
\]

(d) Determine the image of \(\ell\) under inversion relative to the unit circle \(S^1 \subset \mathbb{C}\).