Choose any three of the following seven problems to hand in on Tuesday, June 16.

1. Consider a triangle $\triangle ABC$, and let $x$ be a positive real number. There is a point $L \in BC$ with the property that

$$BL = x(LC)$$

Similarly, there are points $M \in CA$ and $N \in AB$ such that

$$CM = x(MA), \quad AN = x(NB).$$

Show that the lines $\overrightarrow{AL}, \overrightarrow{BM}, \overrightarrow{CN}$ are concurrent if and only if $x = 1$.

2. Suppose $I$ is an isometry. Show that $I$ is injective.

3. (a) Find two rotations $R_1, R_2$ whose composition $R_1 \circ R_2$ is a rotation.

(b) Find two rotations $R_1, R_2$ whose composition is a translation.

(c) Can you find rotations $R_1, R_2$ whose composition is a reflection?

(d) Does the set of all rotations form a group under composition?

4. Suppose that $\ell, \ell'$ are two parallel lines in the Euclidean plane, and let $d$ be the distance between them. Let $R_\ell$ and $R_{\ell'}$ be the reflections over these lines.

(a) Show the following: (i) the composition $R_\ell \circ R_{\ell'}$ is a translation, (ii) the distance of the translation is $2d$, and (iii) the direction of the translation is along the line perpendicular to $\ell'$ and $\ell$.

(b) Is the translation $R_\ell \circ R_{\ell'}$ the same as $R_{\ell'} \circ R_\ell$?

5. Suppose $\triangle ABC$ and $\triangle A'B'C'$ are two congruent triangles in the plane. Explain how to find lines $\ell_1, \ell_2, \ldots, \ell_n$ so that the isometry given by

$$R_{\ell_n} \circ \ldots \circ R_{\ell_2} \circ R_{\ell_1}$$

takes $\triangle ABC$ to $\triangle A'B'C'$. Here $R_\ell$ is the reflection over $\ell$. 


6. (a) Prove the following angle addition formulas for sin and cos. Here \( \alpha, \beta \in \mathbb{R} \).

\[
\begin{align*}
\cos(\alpha + \beta) &= \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \\
\sin(\alpha + \beta) &= \cos(\alpha) \sin(\beta) + \cos(\alpha) \sin(\beta)
\end{align*}
\]

(b) Evaluate the following and use geometry to justify your answers. All angles are measured in radians.

- \( \sin(\pi/3) \)
- \( \sin(\pi/4) \)
- \( \sin(\pi/6) \)

(c) Evaluate \( \sin(\pi/24) \).

7. Suppose \( \theta \in \mathbb{R} \). Explain why \( e^{i\theta} \) lies on the unit circle in \( S^1 \). That is, you should explain why the distance from \( e^{i\theta} \) to the origin is 1.

Extra Credit: Suppose \( A \) is an \( n \times n \) matrix, and define

\[ e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \]

It turns out this series converges to an \( n \times n \) matrix.

(a) Suppose

\[ A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \]

where \( \alpha, \beta \in \mathbb{R} \). Express the matrix \( e^A \) in components. *Hint: Write \( A = \alpha M_1 + \beta M_2 \) for some matrices \( M_1 \) and \( M_2 \) that do not depend on \( \alpha, \beta \).*

(b) Suppose

\[ A = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix} \]

is a diagonal matrix. Express the matrix \( e^A \) in components.

(c) Suppose \( A \) is a diagonal matrix. Show

\[ \det(e^A) = e^{\text{tr}(A)}, \]

where \( \det \) is the determinant and \( \text{tr} \) is the trace.

(d) Show that (1) holds for all \( n \times n \) matrices \( A \).