1. Describe the elements of the set \((\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}\). Is this set countable or uncountable?

2. Let \(A = \{\emptyset, \{\emptyset\}\}\). What is the cardinality of \(A\)? Is \(\emptyset \subset A\)? Is \(\emptyset \in A\)? Is \(\emptyset \in A\)? Is \(\emptyset \in A\)?

3. List the elements of the set \(A \times B\) where \(A\) is the set in the previous question and \(B = \{1, 2\}\).

4. Suppose that \(A, B,\) and \(C\) are sets. Which of the following statements is true for all sets \(A, B,\) and \(C\)? For each, either prove the statement or give a counterexample: \((A \cap B) \cup C = A \cap (B \cup C)\), \(A \cap B \subseteq A \cup B\) if \(A \subseteq B\) then \(A \times A \subset A \times B\), \(A \cap B \cap C = A \cup B \cup C\).

5. State the negation of each of the following statements:
   - There exists a natural number \(m\) such that \(m^3 - m\) is not divisible by 3.
   - \(\sqrt{3}\) is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:
   - (a) Let \(P, Q\) and \(R\) are statements. Then, \(P \land (Q \lor R)\) and \((P \land Q) \lor (P \land R)\) are logically equivalent.
   - (b) Let \(P\) and \(Q\) are statements. Then, \(P \Rightarrow Q\) and \((\sim Q) \Rightarrow (\sim P)\) are logically equivalent.

7. Write the open statement \(P(x, y)\): "for all real \(x\) and \(y\) the value \((x - 1)^2 + (y - 3)^2\) is positive" using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that \(3x + 7\) is odd if and only if \(x\) is even.

9. Prove that if \(a\) and \(b\) are positive numbers, the \(\sqrt{ab} \leq \frac{a + b}{2}\). This is referred to as “Inequality between geometric and arithmetic mean.”

10. Let \(A, B,\) and \(C\) be sets. Prove that \((B \cap C) = (A \times B) \cap (A \times C)\).

11. Let \(A, B,\) and \(C\) be sets. Prove that \((A - B) \cap (A - C) = A - (B \cup C)\).

12. Suppose that \(x\) and \(y\) are real numbers. Prove that if \(x + y\) is irrational, then \(x\) is irrational or \(y\) is irrational.

13. Let \(x\) be an irrational number. Prove that \(x^4\) or \(x^5\) is irrational.

14. Use a proof by contradiction to prove the following.
   - There exist no natural numbers \(m\) such that \(m^2 + m + 3\) is divisible by 4.

15. Let \(a, b\) be distinct primes. Then \(\log_a (b)\) is irrational.

16. Prove or disprove the statement: There exists an integer \(n\) such that \(n^2 - 3 = 2n\).

17. Prove or disprove the statement: There exists a real number \(x\) such that \(x^4 + 2 = 2x^2\).

18. Prove that there exists a unique real number \(x\) such that \(x^3 + 2 = 2x\).

19. Disprove that statement: There exists integers \(a\) and \(b\) such that \(a^2 + b^2 \equiv 3\) (mod 4)

20. Use induction to prove that \(6| (n^3 + 5n)\) for all \(n \geq 0\).

21. Use induction to prove that \(1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2\) for all \(n \in \mathbb{N}\).

22. Use the Strong Principle of Mathematical Induction to prove that for each integer \(n \geq 11\), there are nonnegative integers \(x\) and \(y\) such that \(n = 4x + 5y\).

23. A sequence \(\{a_n\}\) is defined recursively by \(a_0 = 1\), \(a_1 = -2\) and for \(n \geq 1\),
   \[a_{n+1} = 5a_n - 6a_{n-1}\].
   Prove that for \(n \geq 0\),
   \[a_n = 5 \times 2^n - 4 \times 3^n\].

24. Suppose \(R\) is an equivalence relation on a set \(A\). Prove or disprove that \(R^{-1}\) is an equivalence relation on \(A\).

25. Consider the set \(A = \{a, b, c, d\}\), and suppose \(R\) is an equivalence relation on \(A\). If \(R\) contains the elements \((a, b)\) and \((b, d)\), what other elements must it contain?

26. Let \(A = \{a_1, a_2, a_3\}\) and \(B = \{b_1, b_2\}\). Find a relation on \(A \times B\) that is transitive and symmetric, but not reflexive.

27. Suppose \(A\) is a finite set and \(R\) is an equivalence relation on \(A\).
(a) Prove that $|A| \leq |R|$. 
(b) If $|A| = |R|$, what can you conclude about $R$?

28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$R = \{(x \mod 4, 3x \mod 6) \mid x \in \mathbb{Z}\}.$$ 
Prove that $R$ is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. Is $R$ a bijective function?

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

$$S = \{(x \mod 4, 2x \mod 6) \mid x \in \mathbb{Z}\}.$$ 
Prove that $S$ is not a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

30. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that $h$ is bijective.

31. Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if $f$ is injective and $g$ is surjective.

32. (X points) Let $\mathbb{R}^+$ denote the set of positive real numbers and let $A$ and $B$ be denumerable subsets of $\mathbb{R}^+$. Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.

33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.

34. Prove the following statement: A nonempty set $S$ is countable if and only if there exists an injective function $g : S \to \mathbb{N}$.

35. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.

36. Let $a, b, c \in \mathbb{Z}$. Prove that if $c$ is a common divisor of $a$ and $b$, then $c$ divides any linear combination of $a$ and $b$.

37. Define the term “$p$ is a prime”. Then prove that if $a, p \in \mathbb{Z}$, $p$ is prime, and $p$ does not divide $a$, then $\text{gcd}(a, p) = 1$.

38. The greatest common divisor of three integers $a, b, c$ is the largest positive integer which divides all three. We denote this greatest common divisor by $\text{gcd}(a, b, c)$. Assume that $a$ and $b$ are not both zero. Prove the following equation:

$$\text{gcd}(a, b, c) = \text{gcd}(\text{gcd}(a, b), c).$$

39. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n \to \infty} \frac{3n + 1}{n - 2} = 3.$$ 

40. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n \to \infty} \frac{(-1)^n 3n + 1}{n - 2}$$

does not exist.

41. Let $(a_n)$ be a sequence with positive terms such that $\lim_{n \to \infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n \to \infty} \frac{3a_n + 1}{2} = 2.$$ 

42. (a) Use induction to prove

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{4(n + 1)}$$

for all $n \in \mathbb{N}$.

(b) Prove $\sum_{k=1}^{\infty} \frac{1}{2k(2k + 2)} = \frac{1}{4}$. 
