2 Rate of Change: The Derivative

2.1 Instantaneous Rate of Change

* Instantaneous Velocity

The instantaneous velocity of an object at time $t$ is defined to be the limit of the average velocity of the object over shorter and shorter time intervals containing $t$.

Example 1 The distance (in feet) of an object from a point is given $s(t) = t^2$, where time $t$ is in seconds.

(a) What is the average velocity of the object between $t = 2$ and $t = 5$?

(b) By using smaller and smaller intervals around 2, estimate the instantaneous velocity at time $t = 2$.

* Instantaneous Rate of Change

The instantaneous rate of change (IROC) of $f$ at $a$, also called the rate of change of $f$ at $a$, is defined to be the limit of the average rates of change of $f$ over shorter and shorter intervals around $a$.

Example 2 The quantity (in mg) of a drug in the blood at time $t$ (in minutes) is given by $Q = 25(0.8)^t$. Estimate the instantaneous rate of change of the quantity at $t = 3$ and interpret your answer.
* The Derivative at a Point

The **derivative of $f$ at $a$**, written $f'(a)$, is defined to be the instantaneous rate of change of $f$ at the point $a$.

**Example 3**  
*Estimate $f'(3)$ if $f(x) = x^3$.***

* Visualizing the Derivative: Slope of the Graph and Slope of the Tangent Line

The derivative of a function at the point $A$ is equal to the slope of the line tangent to the curve at $A$. 
Example 4 Use a graph of \( f(x) = x^2 \) to determine whether each of the following quantities is positive, negative, or zero: (a) \( f'(1) \) (b) \( f'(-1) \) (c) \( f'(2) \) (d) \( f'(0) \)

Example 5 Estimate the derivative of \( f(x) = 2^x \) at \( x = 0 \) graphically and numerically.
Example 6 The graph of a function \( y = f(x) \) is shown in the following figure. Indicate whether each of the following quantities is positive or negative, and illustrate your answers graphically.

![Graph of a function](image)

(a) \( f'(1) \)
(b) \( \frac{f(3) - f(1)}{3 - 1} \)
(c) \( f(4) - f(2) \)
(d) \( f'(5) \)
(e) \( f'(3) \)

* Estimating the Derivative of a Function Given Numerically

Example 7 The total acreage of farms in the US has decreased since 1980. See the following table.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Farm land (million acres)</td>
<td>1039</td>
<td>1012</td>
<td>987</td>
<td>963</td>
<td>945</td>
</tr>
</tbody>
</table>

(a) What was the average rate of change in farm land between 1980 and 2000?

(b) Estimate \( f'(1995) \) and interpret your answer in terms of farm land.
2.2 The Derivative Function

* Finding the Derivative of a Function Given Graphically

For a function $f$, we define the derivative function, $f'$, by

$$f'(x) = \text{Instantaneous rate of change of } f \text{ at } x.$$

Example 1 Given the graph of $f(x)$ below.

(a) Estimate the derivative of the function $f(x)$ at $x = -2, -1, 0, 1, 2, 3, 4, 5$. Fill in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(b) Plot the values of the derivative function calculated in (a) in the graph below.

* What Does the Derivative Tell Us Graphically?

If \( f' > 0 \) on an interval, then \( f \) is increasing over that interval.
If \( f' < 0 \) on an interval, then \( f \) is decreasing over that interval.
If \( f' = 0 \) on an interval, then \( f \) is constant over that interval.
Example 2 Given the graph of \( f(x) \) below. Sketch the graph of \( f'(x) \).

* Estimating the Derivative of a Function Given Numerically

Example 3 Find approximate values for \( f'(x) \) at each of the x-values given in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>18</td>
<td>13</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td>21</td>
<td>30</td>
</tr>
</tbody>
</table>
* Finding the Derivative of a Function Given by a Formula

**Example 4** Guess a formula for the derivative of \( f(x) = x^2 \).

**Example 5** Draw the graph of a continuous function \( y = f(x) \) that satisfies the following three conditions:

(a) \( f'(x) > 0 \) for \( 0 < x < 2 \)
(b) \( f'(x) < 0 \) for \( x < 0 \) and \( x > 2 \)
(c) \( f'(x) = 0 \) at \( x = 0 \) and \( x = 2 \)
Focus on Theory: Limits and the Definition of the Derivative

* Definition of the Derivative Using Average Rates

For any function \( f \), we define the derivative function, \( f' \), by

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},
\]

provided the limit on the right hand side exists. The function \( f \) is said to be differentiable at any point \( x \) at which the derivative function is defined.

We write

\[
\lim_{x \to c} f(x)
\]

to represent the number approached by \( f(x) \) as \( x \) approaches \( c \).

**Example 1** Investigate

\[
\lim_{x \to 3} x^2
\]

**Example 2** Use a graph to estimate

\[
\lim_{x \to 0} \frac{\sin x}{x},
\]

where \( x \) is in radians.
Example 3 Estimate

\[ \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} \]

numerically.

Example 4 Use algebra to find

\[ \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} \]

* Use the Definition to Calculate Derivatives

Example 5 Show that the derivative of \( f(x) = x^2 \) is \( f'(x) = 2x \).
Example 6 Show that the derivative of \( f(x) = 6x - 4 \) is \( f'(x) = 6 \).

Example 7 Show that the derivative of \( f(x) = x^3 \) is \( f'(x) = 3x^2 \).
2.3 Interpretations of the Derivative

* An Alternative Notation for the Derivative

Given a function \( y = f(x) \), the Leibniz’s notation for the derivative function \( f'(x) \) is

\[
\frac{dy}{dx}
\]

which can be viewed as the derivative with respect to \( x \) of \( y \). And we write

\[
\frac{dy}{dx} \bigg|_{x=a}
\]

to represent \( f'(a) \).

* Using Units to Interpret the Derivative

The units of the derivative of a function are the units of the dependent variable divided by the units of the independent variable. In other words, the units of \( \frac{dA}{dB} \) are the units of \( A \) divided by the units of \( B \).

If the derivative of a function is not changing rapidly near a point, then the derivative is approximately equal to the change in the function when the independent variable increases by 1 unit.

**Example 1** The cost \( C \) (in dollars) of building a house \( A \) square feet in area is given by the function \( C = f(A) \). What are the units and the practical interpretation of the function \( f'(A) \)?

**Example 2** The cost of extracting \( T \) tons of ore from a copper mine is \( C = f(T) \) dollars. What does it mean to say that \( f'(2000) = 100 \)?
Example 3 If \( q = f(p) \) gives the number of thousands of tons of zinc produced when the price is \( p \) dollars per ton, then what are the units and the meaning of

\[
\frac{dq}{dp} \bigg|_{p=900} = 0.2
\]

Example 4 The time, \( L \) (in hours), that a drug stays in a person's system is a function of the quantity administered, \( q \), in mg, so \( L = f(q) \).

(a) Interpret the statement \( f(10) = 6 \). Give units for the numbers 10 and 6.

(b) Write the derivative of the function \( L = f(q) \) in Leibniz notation. If \( f'(10) = 0.5 \), what are the units of the 0.5?

(c) Interpret the statement \( f'(10) = 0.5 \) in terms of dose and duration.

The derivative of velocity, \( \frac{dv}{dt} \), is defined to be acceleration.

Example 5 If the velocity of a body at time \( t \) seconds is measured in meters/sec, what are the units of the acceleration?
* Using the Derivative to Estimate Values of a Function

**Local Linear Approximation**

If \( y = f(x) \) and \( \Delta x \) is near 0, then
\[
\Delta y \approx f'(x) \Delta x.
\]
Then for \( x \) near \( a \) and \( \Delta x = x - a \),
\[
f(x) \approx f(a) + f'(a) \Delta x.
\]
This is called the Tangent Line Approximation.

**Example 6** Fertilizers can improve agricultural production. A Cornell University study on maize (corn) production in Kenya found that the average value, \( y = f(x) \), in Kenyan shillings of the yearly maize production from an average plot of land is a function of the quantity, \( x \), of fertilizer used in kilograms. (The shilling is the Kenyan unit of currency.)

(a) Interpret the statements \( f(5) = 11,500 \) and \( f'(5) = 350 \).

(b) Use the statements in part (a) to estimate \( f(6) \) and \( f(10) \).

* Relative Rate of Change

The relative rate of change (RROC) of \( y = f(t) \) at \( t = a \) is defined to be
\[
\text{RROC of } y \text{ at } a = \frac{dy/dt}{y} = \frac{f'(a)}{f(a)}.
\]
Example 7  Annual world soybean production, $W = f(t)$, in million tons, is a function of $t$ years since the start of 2000.

(a) Interpret the statements $f(8) = 253$ and $f'(8) = 17$ in terms of soybean production.

(b) Calculate the relative rate of change of $W$ at $t = 8$; interpret it in terms of soybean production.

Example 8  Solar photovoltaic (PV) cells are the world’s fastest growing energy source. Annual production of PV cells, $S$, in megawatts, is approximated by $S = 277e^{0.368t}$, where $t$ is in years since 2000. Estimate the relative rate of change of PV cell production in 2010 using

(a) $\Delta t = 1$  
(b) $\Delta t = 0.1$  
(c) $\Delta t = 0.01$
2.4 The Second Derivative

* What Does the Second Derivative Tell Us?

\[ f'' > 0 \text{ on an interval means } f' \text{ is increasing, so the graph of } f \text{ is concave up there.} \]

\[ f'' < 0 \text{ on an interval means } f' \text{ is decreasing, so the graph of } f \text{ is concave down there.} \]

Example 1 Given the graph of the function \( f(x) \) below, determine whether quantities are positive, negative or zero?

![Graph of \( f(x) \)](image)

Fill in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2 Consider the following graph of $y = f(x)$.

(a) Estimate the intervals on which the derivative is positive and the intervals on which the derivative is negative.

(b) Estimate the intervals on which the second derivative is positive and the intervals on which the second derivative is negative.

---

Example 3 For each function given in the following tables, do the signs of the first and second derivatives of the function appear to be positive or negative over the given interval?

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>10.1</td>
<td>11.2</td>
<td>13.7</td>
<td>16.8</td>
<td>21.2</td>
<td>27.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>10.1</td>
<td>9.9</td>
<td>8.1</td>
<td>6.0</td>
<td>3.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>1000</td>
<td>1010</td>
<td>1015</td>
<td>1018</td>
<td>1020</td>
<td>1021</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w(x)$</td>
<td>10.7</td>
<td>6.3</td>
<td>4.2</td>
<td>3.5</td>
<td>3.3</td>
</tr>
</tbody>
</table>
Example 4 Graph the functions described in parts (a)-(d).

(a) First and second derivatives everywhere positive.

(b) Second derivative everywhere negative; first derivative everywhere positive.

(c) Second derivative everywhere positive; first derivative everywhere negative.

(d) First and second derivatives everywhere negative.

Example 5 Sketch a graph of a continuous function $f$ with the following properties:

(a) $f(0) = 1$

(b) $f'(x) > 0$ for all $x$

(c) $f''(x) < 0$ for $x < 0$

(d) $f''(x) > 0$ for $x > 0$

(e) $f'(0) = 1$
2.5 Marginal Cost and Revenue

* Marginal Analysis

The cost function, \( C(q) \), gives the total cost of producing a quantity \( q \) of some good. Define

\[
\text{Marginal Cost} = MC(q) = C'(q),
\]

which gives us that

\[
\text{Marginal Cost} \approx C(q + 1) - C(q).
\]

The revenue function, \( R(q) \), gives the total revenue received from a firm from selling a quantity, \( q \), of some good. Define

\[
\text{Marginal Revenue} = MR(q) = R'(q),
\]

which gives us that

\[
\text{Marginal Revenue} \approx R(q + 1) - R(q).
\]

**Example 1** In the figure below, is marginal cost greater at \( q = 5 \) or at \( q = 30 \)? At \( q = 20 \) or at \( q = 40 \)?

![Graph showing cost function C(q) with points at q=5, q=30, q=20, and q=40]
Example 2 In the figure below, estimate the marginal revenue when the level of production is 750 units and interpret it.

Example 3 For $q$ units of a product, a manufacturer’s cost is $C(q)$ dollars and revenue is $R(q)$ dollars, with $C(500) = 7200$, $R(500) = 9400$, $MC(500) = 15$, and $MR(500) = 20$.

(a) What is the profit or loss at $q = 500$?

(b) If production is increased from 500 to 501 units, by approximately how much does profit change?
Example 4 Let $C(q)$ represent the total cost of producing $q$ items. Suppose $C(1000) = 500$ and $C'(1000) = 25$. Estimate the total cost of producing 1001 items, 999 items and 1100 items.

Example 5 Let $C(q)$ represent the cost and $R(q)$ represent the revenue, in dollars, of producing $q$ items.

(a) If $C(50) = 4300$ and $C'(50) = 24$, estimate $C(52)$.

(b) If $C'(50) = 24$ and $R'(50) = 35$, approximately how much profit is earned by the 51st item?

(c) If $C'(100) = 38$ and $R'(100) = 35$, should the company produce the 101st item? Why or why not?
Example 6 A company's cost of producing $q$ liters of a chemical is $C(q)$ dollars; this quantity can be sold for $R(q)$ dollars. Suppose $C(2000) = 5930$ and $R(2000) = 7780$.

(a) What is the profit at a production level of 2000?

(b) If $MC(2000) = 2.1$ and $MR(2000) = 2.5$, what is the approximate change in profit if $q$ is increased from 2000 to 2001? Should the company increase or decrease production from $q = 2000$?

(c) If $MC(2000) = 4.77$ and $MR(2000) = 4.32$, should the company increase or decrease production from $q = 2000$?